Physica D 277 (2014) 22–39

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

# Networks of diffusively time-delay coupled systems: Conditions for synchronization and its relation to the network topology

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## HIGHLIGHTS

- Conditions for boundedness of solutions of networks of time-delay coupled systems.
- Conditions for local and global synchronization of time-delay coupled systems.
- All synchronization conditions are related to the network topology.

## ARTICLE INFO

*Article history:*
Received 30 December 2013
Received in revised form
4 February 2014
Accepted 6 March 2014
Available online 24 March 2014
Communicated by A. Pikovsky

*Keywords:*
Time-delayed diffusive coupling
Bounded solutions
Synchronization
Network topology

## ABSTRACT

We consider networks of time-delayed diffusively coupled systems and relate conditions for synchronization of the systems in the network to the topology of the network. First we present sufficient conditions for the solutions of the time-delayed coupled systems to be bounded. Next we give conditions for local synchronization and we show that the values of the coupling strength and time-delay for which there is local synchronization in any network can be determined from these conditions. In addition we present results on global synchronization in relation to the network topology for networks of a class of nonlinear systems. We illustrate our results with examples of synchronization in networks with FitzHugh–Nagumo model neurons and Hindmarsh–Rose neurons.



## 1. Introduction

Synchronization of interacting dynamical systems has a lot of interesting applications in fields varying from biology to engineering, see, for instance, [1–3] and the references therein. We consider synchronization in networks of systems that interact via *time-delayed diffusive coupling*, where synchronization is understood as the asymptotic matching of the states of all systems in the network. Diffusive coupling is a linear coupling determined by the weighted difference of the outputs of connected nodes. It is found in, for instance, networks of coupled conductance-based model neurons [4–11], where diffusive coupling models electrical synapses [8], networks of biological systems [12–14], coupled mechanical systems [15–19] and electrical systems [20,21]. A time-delay is included in the coupling to take account of the amount of time it takes to exchange and process information. For instance, due to the finite propagation speed of the membrane potential through the neuron's axon [22], the interaction between two coupled neurons is not instantaneous. Another example is when humans are trying to drive their cars at a fixed distance of each other by comparing the distance and/or velocity between their vehicle and the vehicle ahead. If the distance or velocity changes the driver has to decide whether to accelerate or decelerate. Experiments and simulator results have shown that the total reaction time of the driver, that is, the sum of the time it takes to receive and process visual information, the time needed to make a decision and the time it takes to hit the brake or the accelerator pedal, varies between 0.6 s and 2 s, [23].

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http://dx.doi.org/10.1016/j.physd.2014.03.004
0167-2789/

The time-delayed diffusive coupling considered in this paper is of the form

$$u_i(t) = \sigma \sum_{j\in\mathcal{N}_i} a_{ij}[y_j(t-\tau) - y_i(t-\tau)].$$

Here $y_i(t), y_j(t)$ are the outputs of systems $i$ and $j$, respectively, $u_i(t)$ is the input for system $i$, $\sigma$ is the coupling strength, $a_{ij}$ are the interaction weights representing the weighted communication structure of the network and $\tau$ is the time-delay. The set $\mathcal{N}_i$ is the neighbor set of system $i$, which specifies the systems that connect to system $i$. It is assumed that the network structure does not change in time, that is, $\mathcal{N}_i$ does not depend on time.

The coupling we consider is a particular type of time-delayed diffusive coupling. Another relevant type of time-delayed diffusive coupling is of the form

$$u_i(t) = \sigma \sum_{j\in\mathcal{N}_i} a_{ij}[y_j(t-\tau) - y_i(t)].$$

Note that in this type of coupling the time-delay appears only in the received outputs, which has the consequence that in general the coupling term does not vanish when the systems are synchronized. This means that the synchronized solutions of the coupled system will depend on the values of the coupling strength, time-delay and network structure, which will obviously complicate matters. We will therefore only consider the former type of time-delayed diffusive coupling. However, we will briefly comment on the latter type of coupling in the light of our results in the discussion section.

We will present conditions for local and global synchronization in networks with systems that interact via the first type of time-delayed diffusive coupling. The difference between local and global synchronization is that in the former type of synchronization the trajectories of the systems are assumed to be sufficiently near each other, whereas in the latter no such assumption on the initial data is made. Local synchronization in such networks is considered in, for instance, [24–27]. In these references a *Master Stability Function* (MSF) [28] for the coupled system is derived and with the help of this MSF conditions for *local* stability of the synchronization manifold of the coupled systems are determined. In particular the results in [26,27] show that synchronization in a network with chaotic systems where the trajectory of an uncoupled system is also a solution of the network (as is the case in the present paper) is not possible if the time-delay is too large. A nice advantage of this approach is that the condition for stability of the MSF gives conditions for synchronization in any network. However, a disadvantage is that these conditions are necessary for synchronization, but generally not sufficient. For zero time-delay, [29] presents examples that show that the MSF approach might fail if the uncoupled systems does not have an attractor. However, assuming that the uncoupled systems have an attractor is not sufficient to conclude that the systems locally synchronize via the MSF approach. It is known that negative Lyapunov exponents, the criteria often used for stability of the MSF, do not necessarily imply stability of the linear time-varying variational system [30]. In particular, for zero time-delay, in [31,32] it is shown that the dynamics near the synchronization manifold of coupled chaotic oscillators with an asymptotically stable attractor might produce a specific type of intermittent behavior associated with a temporal loss of synchrony. This phenomenon is called *attractor bubbling* and may occur despite the Lyapunov exponents of the linear variational equation all being negative.

Our results regarding local synchronization are somewhat related to the MSF approach in the sense that the conditions for synchronization follow from the stability properties of a linear variational system. However, we do impose additional constraints such that our conditions are sufficient to conclude local synchronization. In particular we will show how to construct a *local synchronization diagram* of a network of two systems, which is a non-empty set consisting of the values of $\sigma$ (the coupling strength) and $\tau$ (the time-delay) for which the zero solution of the linear variational system is uniformly asymptotically stable. This local synchronization diagram acts as a template from which the values of $\sigma$ and $\tau$ can be determined for which a arbitrary network of time-delay diffusively coupled systems locally synchronizes. More precisely, the values of $\sigma$ and $\tau$ for which the network locally synchronizes are determined by the intersection of scaled copies of the local synchronization diagram, where the scaling factors depend only on the eigenvalues of the weighted Laplacian matrix of the network. A clear advantage of this approach is that it is computationally efficient since, once we have determined the local synchronization diagram, we can give a sufficient condition for local synchronization in any network, simply by computing the eigenvalues of the Laplacian and scaling.

Results for global synchronization of diffusively time-delay coupled systems are presented in, for instance, [33,34]. The results in these references are consistent with the results of [26,27] in the sense that synchronization is only guaranteed if the time-delay does not exceed some positive threshold (which is proven to exist under appropriate conditions). In addition, the results presented in [34] also provide some insight in the conditions for synchronization in relation to the structure, or topology, of the network. However, these results only apply to some specific network topologies. To the best of our knowledge, there are no results that explicitly relate (global) synchronization of systems and the topology of the network in the general case. However, in [35] experimental evidence is presented for the existence of a general relationship between the topology of the network and synchronization. Based on the result in [33], we show that there exist values of $\sigma$ and $\tau$ for which two diffusively time-delay coupled systems globally synchronize. This result implies the existence of a *global synchronization diagram*. We show that conditions for global synchronization in any network are determined by the intersection of two scaled copies of the global synchronization diagram where the scaling factor of one copy is the smallest non-zero eigenvalue of the Laplacian, while the scaling factor of the other copy is the largest eigenvalue of the Laplacian.

The paper is organized as follows. After formally introducing the problem setting and some additional notation, we present, in Section 2, conditions for boundedness of solutions of the coupled system. Knowing that the solutions of the coupled systems are bounded is important since our notion of synchronization is an asymptotic one. Sections 3 and 4 present conditions to construct the local, respectively global, synchronization diagram and show how to determine the values for $\sigma$ and $\tau$ for which the network synchronizes. Section 6 concludes the paper with a summary of our results and a discussion.

### *1.1. Problem setting*

We consider systems of the form

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + Bu_i(t) \\ y_i(t) = Cx_i(t) \end{cases} \tag{1}$$

with $i = 1, 2, \dots, k$, state $x_i(t) \in \mathbb{R}^n$, inputs $u_i(t) \in \mathbb{R}^m$ and outputs $y_i(t) \in \mathbb{R}^m$, $1 \le m \le n$, sufficiently smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$ and matrices $B$ and $C$ of appropriate dimension. It is assumed that $CB$ is similar [36] to a positive definite matrix. For notational convenience we only consider the case that $CB = I_m$, with $I_m$ the $m \times m$ identity matrix.

We let the communication structure be defined by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{1, 2, \dots, k\}$ the set of nodes and $\mathcal{E}$ the set of edges. We consider only *undirected graphs*, i.e., for every $i, j \in \mathcal{V}$, if $(i, j) \in \mathcal{E}$, then $(j, i) \in \mathcal{E}$. We assume that the graph does not have any self-connections, i.e., $(i, i) \notin \mathcal{E}$ for all $i \in \mathcal{V}$. Also the graph $\mathcal{G}$ is assumed to be *connected*, that is, for every two nodes $i, j \in \mathcal{V}$, there exists a path (a sequence of edges) between $i$ and $j$. The *set of neighbors* of node $i \in \mathcal{V}$, denoted by $\mathcal{N}_i$, is defined as

$$\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}.$$

The interaction between systems (1) is governed by the following *time-delayed diffusive coupling* law:

$$u_i(t) = \sigma \sum_{j\in\mathcal{N}_i} a_{ij}[y_j(t-\tau) - y_i(t-\tau)]. \tag{2}$$

Here the positive constant $\sigma$ is the *coupling strength*, positive constants $a_{ij}$ are the *interconnection weights* and non-negative constant $\tau$ represents a *time-delay*. It is assumed that $a_{ij} = a_{ji}$. Moreover, the fact the presence of the coupling strength $\sigma$ in (2) implies that we can assume without loss of generality that $\max_{i\in\mathcal{V}} \sum_{j\in\mathcal{N}_i} a_{ij} = 1$.

We let the matrix $A$ have entries $a_{ij}$ if $(i, j) \in \mathcal{E}$ and 0 otherwise. Note that the assumptions that $\mathcal{G}$ is undirected and $a_{ij} = a_{ji}$ imply that $A$ is symmetric, $A = A^\top$, where $^\top$ denotes transposition. The matrix $A$ is known as the *weighted adjacency matrix*. We let $D$ be the *weighted degree matrix*, which is the diagonal matrix with entries $d_i = d_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$, and we define the *weighted Laplacian matrix* $L$ by the relation $L = D - A$. Clearly $L$ is symmetric and, as it has zero row sums, it is singular. Because the multiplicity of the zero eigenvalue equals the number of connected components of the graph, cf. [37], the assumption of $\mathcal{G}$ being connected implies that zero is a simple eigenvalue of $L$. Moreover, since $L = L^\top$ all eigenvalues of $L$ are real valued, and an application of Geršgorin's Disc theorem, cf. [36], shows that all eigenvalues are non-negative. Thus we can order the eigenvalues $\lambda_i$ of $L$ as

$$0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_k.$$

Note that by the assumption that $\max_{i\in\mathcal{V}} d_i = 1$ and again Geršgorin's Disc Theorem, cf. [36], we have $\lambda_k \le 2$.

Let $r$ be a non-negative integer and let $\mathcal{C}^r(\mathcal{X}, \mathcal{Y})$ denote the space of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$ that are at least $r$ times continuously differentiable. For $r = 0$ we write $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ instead of $\mathcal{C}^0(\mathcal{X}, \mathcal{Y})$. Following [38], we let $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^{kn})$ be the phase–space of the coupled system (1), (2). For an element $x_t \in \mathcal{C}$, we denote, as usual, $x_t(\theta) = \{x(t+\theta), \theta \in [-\tau, 0]\}$, $t \in \mathbb{R}$, with $x(t) = \operatorname{col}(x_1(t), x_2(t), \ldots x_k(t))$. The space $\mathcal{C}$ is equipped with norm $\|\cdot\|$, $\|\phi\| = \sup_{-\tau\le\theta\le 0} |\phi(\theta)|$ for $\phi \in \mathcal{C}$, where $|\cdot|$ is a vector norm in $\mathbb{R}^{kn}$. In this paper we let throughout $|\cdot|$ be the Euclidean norm. A solution of the coupled system (1), (2) through $(t_0, \phi) \in \mathbb{R}\times\mathcal{C}$ is a function $x_t(t_0, \phi)$ defined for $t \in [t_0, t_0+T]$ for some $T \in \mathbb{R}_+ = (0, \infty)$ which satisfies (1), (2) with $x_{t_0}(t_0, \phi) = \phi$. The notation $x(t; t_0, \phi)$ stands for $x_t(0) = x_t(t_0, \phi)(0)$ for a solution $x_t \in \mathcal{C}$. Since we assume the function $f$ (in (1)) to be sufficiently smooth we know that solutions of (1), (2) exist, at least locally, and that they are unique. For our purpose, i.e. synchronization in a network of systems, we require the solutions of the coupled system to exist for all $t \ge t_0$, hence we present in the next section conditions for the coupled system to be uniformly (ultimately) bounded. These terms are defined below.

**Definition 1** (*Uniform Boundedness and Uniform Ultimate Boundedness, [39]*)**.** The solutions of the coupled system (1), (2) are uniformly bounded at $t = t_0$, or simply uniformly bounded, if for any $B_1 > 0$, there is a $B_2 > 0$ such that $\phi \in \mathcal{C}$, $\|\phi\| \le B_1$, implies $|x(t; t_0, \phi)| < B_2$ for $t \ge t_0$. The solutions of the coupled system (1), (2) are uniformly ultimately bounded for bound $B_3$ at $t = t_0$, or simply, uniformly ultimately bounded, if for each $B_4 > 0$ there is a $K > 0$ such that, $\phi \in \mathcal{C}$, $\|\phi\| \le B_4$, $t \ge K$ imply that $|x(t; t_0, \phi)| < B_3$.

For $M > 0$ we define $\mathcal{C}_M = \{\phi \in \mathcal{C} | \|\phi\| \le M\}$. Whenever the solutions of the coupled system exist and are bounded with $x_t \in \mathcal{C}_M$ for all $t \ge t_0$, we define *local synchronization* and *global synchronization* as follows:

**Definition 2** (*Local Synchronization and Global Synchronization*)**.** Let $M > 0$ be given and assume that the set $\mathcal{C}_M$ is a forward invariant set for the coupled system (1), (2). Systems (1) with coupling (2) locally synchronize if there is a $\delta > 0$ such that, for all $i, j$, $\|\phi_i - \phi_j\| \le \delta$ implies $\lim_{t\to\infty} |x_i(t; t_0, \phi) - x_j(t; t_0, \phi)| = 0$. Systems (1) with coupling (2) globally synchronize if, for all $i, j$ and for all $\phi$, $\lim_{t\to\infty} |x_i(t; t_0, \phi) - x_j(t; t_0, \phi)| = 0$.

### 1.2. Additional notation

We let $\mathbb{R}_+$ be the set of non-negative real numbers. A continuous function $w : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $w(0) = 0$. The symbol $\mathcal{K}_\infty$ denotes the class of functions that belong to $\mathcal{K}$ with the additional property that $w(s) \to \infty$ as $s \to \infty$. A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is called positive (semi-)definite, denoted by $P > 0$ ($P \ge 0$), if its associated quadratic form $x^\top Px$ is a positive (semi-)definite function. When no confusion can arise, we denote by $\|\cdot\|$ the norm of a matrix induced by the vector norm $|\cdot|$, i.e., for $Q \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, $\|Q\| := \max_{|x|=1} |Qx|$. Given two matrices $P = (p_{ij}) \in \mathbb{R}^{k\times\ell}$, $Q \in \mathbb{R}^{m\times n}$, the Kronecker product $P \otimes Q$ is the $km \times \ell n$ matrix defined by

$$P \otimes Q = \begin{pmatrix} p_{11}Q & \cdots & p_{1\ell}Q \\ \vdots & \ddots & \vdots \\ p_{k1}Q & \cdots & p_{k\ell}Q \end{pmatrix}.$$

The notation $\mathbf{1}_k$ ($\mathbf{0}_k$) stands for the $k$-dimensional vector with all entries equal to 1 (0).

For $a \in \mathbb{R}^n$, $R \in \mathbb{R}_+$, we define the ball with radius $R$ that is centered at $a$ by

$$\mathcal{B}(a, R) = \{x \in \mathbb{R}^n \mid |x - a| < R\}.$$

## 2. Bounded solutions of the coupled system

Our notion of synchronization requires trajectories of the coupled system (1), (2) to coincide as $t \to \infty$, hence this is an asymptotic notion. Therefore it is necessary to ensure that solutions of the coupled system are well-defined (i.e., exist and are bounded) on $[t_0 - \tau, \infty)$. There are obvious cases in which solutions of the coupled system are well-defined, e.g. if $x_i(t)$ represent phase-dynamics, but having bounded solutions is not trivial. For instance, in [40] it was shown that solutions of $k = 2$ systems (1) with

$$f(x_i(t)) = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -4 & 2 & -3 \end{pmatrix} x_i(t) =: Ax_i(t), \qquad B = C^\top = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

with coupling (2) with $\tau = 0$ grow unbounded when $\sigma > 0.6513$, even though the zero solution of the uncoupled system $\dot{s}(t) = As(t)$ is globally asymptotically stable. It can easily be shown that there are non-zero values for $\tau$ for which the solutions of these systems grow unbounded too.

We present conditions for solutions of the coupled system to be bounded based on the notion of strictly semi-passive systems. An advantage of this approach is that these conditions only depend on the dynamics of the systems (1), i.e. on the function $f$ and matrices $B$ and $C$, and do not involve the network topology directly. It was shown in [40] that many models that describe the dynamical fluctuations of the membrane potential of neurons have this strict semi-passivity property, while it was shown in [41] that the Lorenz systems is strictly semi-passive. However, assuming the systems (1) to have such property obviously restricts the class of systems under consideration. The conditions for synchronization that we present in the next sections will of course hold for systems which are not strictly semi-passive but for which the solutions of the coupled system (1), (2) are bounded. General theorems for boundedness of solutions of the delay-differential equations can be found in, for instance, [42,39] (via Lyapunov functionals) and [43,38] (via the Razumikhin method).

We define strict semi-passivity as follows.

**Definition 3** (*Strict Semi-passivity*)**.** Consider a system

$$\begin{cases}\dot{x}(t) = f(x(t), u(t)),\\ y(t) = h(x(t)),\end{cases} \tag{3}$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^m$ and sufficiently smooth functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$. Let there exist a storage function $S \in \mathcal{C}^r(\mathbb{R}^n, \mathbb{R}_+)$, $r \geq 1$, and functions $s_1, s_2 \in \mathcal{K}_\infty$ such that along orbits of (3) one has that

$$s_1(|x(t)|) \leq S(x(t)) \leq s_2(|x(t)|) \tag{4}$$

and

$$\dot{S}(x(t)) \leq -H(x(t)) + (y^\top u)(t), \tag{5}$$

with a continuous function $H : \mathbb{R}^n \to \mathbb{R}$. The system (3) is called strictly semi-passive if there is a constant $R > 0$ and a function $s_3 \in \mathcal{K}_\infty$ such that (5) holds with function $H(x_i(t)) \geq s_3(|x_i(t)|)$ for all $|x_i(t)| \geq R$.

**Theorem 1.** *Suppose that each of the uncoupled systems (1) is strictly semi-passive. Then there exists a constant $\sigma_{\max} > 0$ such that the solutions of the coupled systems (1), (2) are uniformly bounded and uniformly ultimately bounded for $\sigma \in [0, \sigma_{\max}]$ and any finite $\tau$.*

**Proof.** Let $\phi = \operatorname{col}(\phi_1, \ldots, \phi_k) \in \mathcal{C}$ and define

$$\begin{aligned} V(\phi) &= S(\phi_1(0)) + \cdots + S(\phi_k(0)) \\ &\quad + \sigma \int_{-\tau}^{0} \phi^\top(s)(I_k \otimes C^\top C)\phi(s)\mathrm{d}s. \end{aligned}$$

Then, invoking the strict semi-passivity property,

$$\begin{aligned} \dot{V}(\phi) &\leq -H(\phi_1(0)) - \cdots - H(\phi_k(0)) - \sigma\phi(0)^\top \\ &\quad \times (L \otimes C^\top C)\phi(-\tau) + \sigma\phi^\top(0)(I_k \otimes C^\top C)\phi(0) \\ &\quad - \sigma\phi^\top(-\tau)(I_k \otimes C^\top C)\phi(-\tau) \\ &= -(H(\phi_1(0)) - 2\sigma\,|C\phi_1(0)|^2) - \cdots - (H(\phi_k(0)) \\ &\quad - 2\sigma\,|C\phi_1(0)|^2) - \sigma\phi(0)^\top(L \otimes C^\top C)\phi(-\tau) \\ &\quad - \sigma\phi^\top(0)(I_k \otimes C^\top C)\phi(0) - \sigma\phi^\top(-\tau)(I_k \otimes C^\top C)\phi(-\tau) \\ &\quad - (H(\phi_1(0)) - 2\sigma\,|C\phi_1(0)|^2) \\ &\quad - \cdots - (H(\phi_k(0)) - 2\sigma\,|C\phi_1(0)|^2) \\ &\quad - \sigma \begin{pmatrix}\phi(0)\\ \phi(-\tau)\end{pmatrix}^\top \left( \begin{pmatrix} I_k & \frac{1}{2}L \\ \frac{1}{2}L & I_k \end{pmatrix} \otimes C^\top C \right) \begin{pmatrix}\phi(0)\\ \phi(-\tau)\end{pmatrix}. \end{aligned}$$

It is easy to verify, using the Schur complement, cf. [36], and the fact that $\lambda_k \leq 2$, that the symmetric matrix

$$\begin{pmatrix} I_k & \frac{1}{2}L \\ \frac{1}{2}L & I_k \end{pmatrix} \geq 0,$$

hence

$$\begin{aligned} \dot{V}(\phi) &\leq -(H(\phi_1(0)) - 2\sigma\,|C\phi_1(0)|^2) - \cdots \\ &\quad - (H(\phi_k(0)) - 2\sigma\,|C\phi_1(0)|^2). \end{aligned}$$

Since $s_3 \in \mathcal{K}_\infty$ there exist two constants $\kappa > 0$ and $R_1 > R$ such that $H(x_i(t)) - \kappa\,|Cx_i(t)|^2 \geq s_3(|x_i(t)|)$ for all $|x_i(t)| \geq R_1$. This implies that there exists a positive constant $M$ such that, for $0 \leq \sigma \leq \sigma_{\max} =: \frac{\kappa}{2}$ such that

$$\dot{V}(\phi) \leq M - s_3(|\phi_1(0)|) - \cdots - s_3(|\phi_k(0)|) =: -w_4(|\phi(0)|) + M.$$

Note that

$$w_1(|\phi(0)|) \leq V(\phi) \leq w_2(|\phi(0)|) + w_3\left(\int_{-\tau}^{0} w_4(|\phi(s)|)\mathrm{d}s\right),$$

with

$$w_3(r) = \frac{\kappa}{2} s_3^{-1}(r),$$

where $w_1(r)$ and $w_2(r)$ depend on the functions $s_1(r)$ and $s_2(r)$, respectively. Applying Theorem 4.2.10 of [39] (also provided in Appendix B) then gives the result.

**Corollary 2.** *Suppose that each system (1) is strictly semi-passive and there is a constant $\kappa > 0$ such that $H(x_i(t)) \geq s_3(|x_i(t)|) + \kappa\,|Cx_i(t)|^2$ for all $|x_i(t)| \geq R$. The solutions of (1), (2) are uniformly bounded and uniformly ultimately bounded for $\sigma \in [0, \frac{\kappa}{2}]$ and any finite $\tau$.*

**Example 1** (*Bounded and Unbounded Solutions of Coupled Lorenz Systems*)**.** Consider system (1) with

$$\begin{aligned} f(x_i(t)) &= \begin{pmatrix} a(x_{i,2}(t) - x_{i,1}(t)) \\ bx_{i,1}(t) - x_{i,2}(t) - x_{i,1}(t)x_{i,3}(t) \\ -cx_{i,3}(t) + x_{i,1}(t)x_{i,2}(t) \end{pmatrix}, \\ B &= C^\top = \begin{pmatrix}1\\0\\0\end{pmatrix}, \end{aligned} \tag{6}$$

with constant parameters $a, b, c > 0$. System (6) is the well-known Lorenz system [44] for which it was shown in [41] that this system is strictly semi-passive with the given matrices $B$ and $C$ in (6). Indeed, consider the positive definite storage function

$$S(x_i(t)) = \tfrac{1}{2}\left(x_{i,1}^2(t) + x_{i,2}^2(t) + (x_{i,3}(t) - a - b)^2\right).$$

A straightforward computation shows that $\dot{S}(x_i(t)) \leq (y_i u_i)(t) - H(x_i(t))$ with the function

$$H(x_i(t)) = ax_{i,1}^2(t) + x_{i,2}^2(t) + cx_{i,3}(t)\left(x_{i,3}(t) - a - b\right),$$

which is non-negative outside the ball $\mathcal{B}$ centered around $(0, 0, a+b)$ with radius $R = a + b$. Note that at the center of $\mathcal{B}$, the point $(0, 0, a+b)$, we have $S = 0$, hence the Lorenz system satisfies the definition of strict semi-passivity after the change of coordinates $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - a - b)$.

For two coupled Lorenz systems and $\sigma \in [0, \sigma_{\max}]$, $\sigma_{\max} < \frac{a}{2}$, Corollary 2 guarantees that solutions are uniformly bounded and uniformly ultimately bounded. Fig. 1 shows simulations results (the outputs of both systems) for $a = 10, b = 28, c = \frac{8}{3}, \tau = 1$ and initial data $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^6)$, $\phi(\theta) = \operatorname{col}(0, 0, 0, 1, 1, 1)$ for $-1 \leq \theta \leq 0$. As can be seen in Fig. 1, and as expected, the outputs of the two coupled Lorenz systems remain bounded for $\sigma = 4.5$. Fig. 1(b) shows that for this $\phi$ the outputs grow unbounded when $\sigma = 5.5$. △

**Example 2** (*Bounded Solutions of Two Coupled FitzHugh–Nagumo Model Neurons*)**.** Consider system (1) with

$$f(x_i(t)) = \begin{pmatrix} x_{i,1}(t) - \frac{1}{3}x_{i,1}^3(t) - x_{i,2}(t) \\ \frac{8}{100}\left(x_{i,1}(t) - \frac{8}{10}x_{i,2}(t)\right) \end{pmatrix}, \quad B = C^\top = \begin{pmatrix}1\\0\end{pmatrix}. \tag{7}$$

System (7) is the FitzHugh–Nagumo model neuron [45,46] which models the membrane potential of a neuron (the output $x_{i,1}(t)$) as a function external stimulus (input $u_i(t)$). Is was shown in [40] that

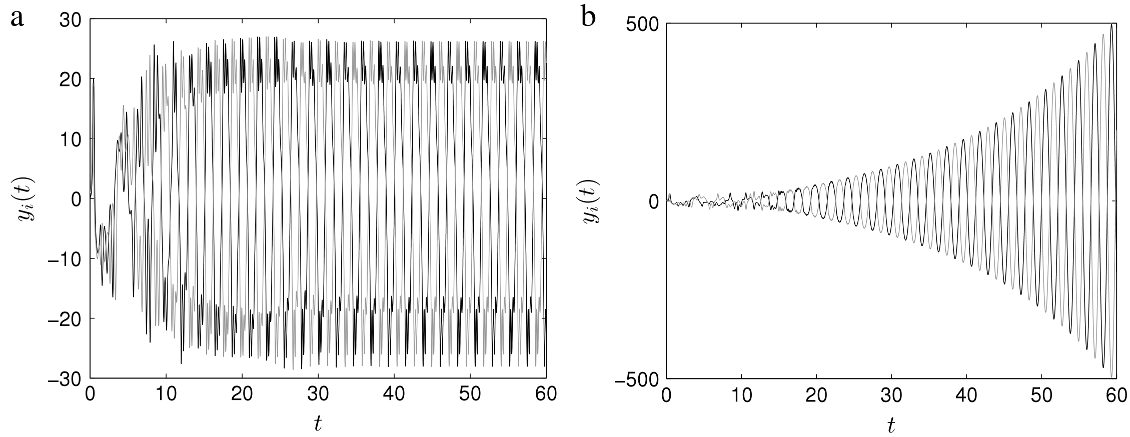

**Fig. 1.** Outputs of two Lorenz systems (6) coupled via (2) with $a_{12} = a_{21} = 1$ and $\tau = 1$. (a) Bounded solutions for $\sigma = 4.5$; (b) solutions may grow unbounded for $\sigma = 5.5$.

system (7) is strictly semi-passive with the given input and output with storage function

$$S(x_i(t)) = \frac{1}{2}\left(x_{i,1}^2(t) + \tfrac{100}{8}x_{i,2}^2(t)\right). \tag{8}$$

Indeed, a simple computation shows that

$$\dot{S}(x_i(t)) \le -H(x_i(t)) + y_i u_i(t)$$

with

$$H(x_i(t)) = x_{i,1}^2(t)\left(\tfrac{1}{3}x_{i,1}^2(t) - 1\right) + \tfrac{8}{10}x_{i,2}^2(t).$$

Clearly the function $H$ is positive for $\left|x_{i,1}(t)\right|$ sufficiently large. In particular, $H(x_i(t)) - \kappa\,|Cx_i(t)|^2$ is positive for every finite $\kappa$ if $|x_i(t)|$ is sufficiently large. This proves, via Corollary 2, uniform boundedness and uniform ultimate boundedness for every finite $\sigma$. We will now compute a bound on the solutions for a network with $k = 2$ systems. This bound will be used in Example 3 below and Example 5 in Section 4. We compute the bound with the help of Remark 4 in the appendix.

We will now compute a bound on the solutions for a network with $k = 2$ systems. This bound will be used in Example 3 below and Example 5 in Section 4. We compute the bound with the help of Remark 4 in the Appendix. Note that even though the conditions for uniform boundedness and uniform ultimate boundedness are independent of the value of the delay, the bound itself does depend on the value of the delay. Hence we have to choose a maximum value for the delay. In addition we have to pick a value for $\kappa$. Note that the maximum delay $\tau_{\max}$ and $\kappa$ are not unique. For the sake of illustration we therefore make a choice of $\tau_{\max} = 0.4$ and $\kappa = 4$. Then $M = \frac{6}{4}(1 + 0.8 + \kappa)^2 = 50.46$. With

$$w_1(r) = \tfrac{1}{2}r^2, \qquad w_2(r) = \tfrac{25}{4}r^2,$$
$$w_3(r) = \tfrac{\kappa}{2}\tfrac{10}{8}r, \qquad w_4(r) = \tfrac{8}{10}r^2$$

we have, for $\sigma \le \frac{\kappa}{2} = 2$, if $\|\phi\| = \|\mathrm{col}\,(\phi_1, \phi_2)\| \le B_1 = 7.8$, then $|x(t)| = |\mathrm{col}\,(x_1(t), x_2(t))| < B_2 = 30$ for all $t \ge t_0$. △

In absence of delay in the coupling, i.e. coupling of the form

$$u_i(t) = \sigma \sum_{j\in\mathcal{N}_i} a_{ij}(y_j(t) - y_i(t)), \tag{9}$$

the semi-passivity property is sufficient to guarantee uniform ultimate boundedness (and thus uniform boundedness since in this case the coupled system is an ordinary differential equation) for any $\sigma$, cf. [41]. There is thus a gap between this result and the result stated in Theorem 1, which may be explained by the fact that the conditions in Theorem 1 are independent of the value of the delay. The following delay-dependent conditions partly bridge the gap.

**Theorem 3.** *Suppose that system (1) is strictly semi-passive. Then there exist positive constants $B_1, B_2, B_2 > B_1 > R$, and constants $\tau_{\max}$ and $\rho = \rho(B_1, B_2, \tau_{\max}) > 0$, such that if $\tau \le \tau_{\max}$ and $\sigma\tau \le \rho$, then all solutions $x(t; t_0, \phi)$ of the coupled system (1), (2) satisfy $|x(t; t_0, \phi)| < B_2$ for all $t \ge t_0$.*

The proof of Theorem 3 with estimates for $\tau_{\max}$ and $\rho$ is provided in Appendix B.

**Example 3** (*Bounded Solutions of Two Coupled FitzHugh–Nagumo Model Neurons, Continued*)**.** In Example 2 we showed that the solutions of two coupled FitzHugh–Nagumo model neurons with initial data $\|\phi\| \le B_1 = 7.8$ remain bounded with bound $B_2 = 30$ as long as $\sigma \le \frac{\kappa}{2} = 2$ and $\tau \le \tau_{\max} = 0.4$. We continue this example and we determine, with the help of Theorem 3, the numbers $B^*$, $\rho$ and a new $\tau_{\max}$ such that if $\sigma\tau \le \rho$, $\tau \le \tau_{\max}$ and $\|\phi\| \le B_1 = 7.8$, then $|x(t)| < B_2 = 30$ for all $t \ge t_0$. With $B^* = 7.9$ and $\kappa = 4$ we find $\rho = 0.001$ and $\tau_{\max} = \frac{1}{100}\rho$. However, this result looks extremely conservative. The conservatism is mainly due to the fact that $|f(x_i(t))|$ is proportional to $|x_i(t)|^3$ for large $|x_i(t)|$, hence the terms in $F(\cdot)$ grow large. However, this is mainly an artifact of the model as in practice large values of $x_i$ will not occur. Therefore, it is justified to rather consider a "saturated" version of the model in such a way that the original model has the same attractor as the saturated version. For example, consider a slightly modified system

$$f(x_i(t)) = \begin{pmatrix} x_{i,1}(t)\left(1 - \frac{1}{3}\psi(x_{i,1})\right) - x_{i,2}(t) \\ \frac{8}{100}\left(x_{i,1}(t) - \frac{8}{10}x_{i,2}(t)\right) \end{pmatrix}, \qquad B = C^\top = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{10}$$

where $\psi$ is smooth and satisfies

$$\psi(x) = \begin{cases} x^2 & (|x| < \sqrt{10} - \epsilon) \\ 10 & (|x| \ge \sqrt{10}) \end{cases}$$

for $\epsilon$ sufficiently small. This system can be shown to have the same attractor as (7), and for this system we find that solutions with initial data $\|\phi\| \le 7.8$ satisfy $|x(t)| < 30$ for $\tau \le \tau_{\max} = 0.01$ and $\sigma\tau \le \rho = 0.01$. △

## 3. Conditions for local synchronization

Let $x(t) = \mathrm{col}\,(x_1(t), \ldots, x_k(t))$ and $F(x(t)) = \mathrm{col}\big(f(x_1(t)), \ldots, f(x_k(t))\big)$. We can then write the coupled system (1), (2) as

$$\dot{x}(t) = F(x(t)) - \sigma(L \otimes BC)x(t - \tau). \tag{11}$$

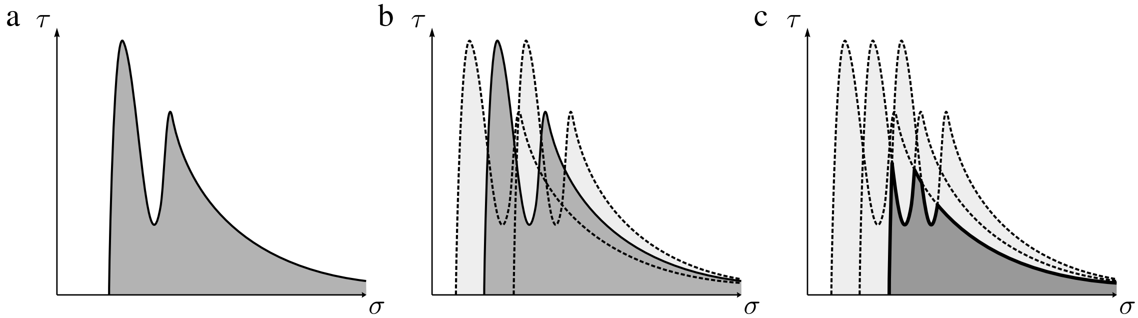


**Fig. 2.** A graphical representation of Lemma 4 for $k = 4$ systems and $L$ having non-zero eigenvalues $\lambda_2 < \lambda_3 = 1 < \lambda_4$: (a) the region $\mathcal{S} \cap \mathcal{S}_M$ (shaded area), (b) scaled copies $\mathcal{S}_2$ and $\mathcal{S}_4$ (light shades) and $\mathcal{S}_3 = \mathcal{S}$ (dark shade), and (c) the dark shaded area determined by the intersections of scaled copies of $\mathcal{S}$ gives values of $\sigma$ and $\tau$ for which local synchronization is guaranteed.

Choosing $\tilde{x}_j(t) = x_j(t) - x_k(t), j = 1, \ldots, k-1$, we can derive from (11) the synchronization error system

$$\dot{\tilde{x}}(t) = \tilde{F}(t, \tilde{x}(t)) - \sigma(\tilde{L} \otimes BC)\tilde{x}(t-\tau), \tag{12}$$

with $\tilde{x}(t) = \mathrm{col}\left(\tilde{x}_1(t), \ldots, \tilde{x}_{k-1}(t)\right)$,

$$\tilde{F}(t, \tilde{x}(t)) = \mathrm{col}\big(f((\tilde{x}_1 + x_k)(t)) - f(x_k(t)), \ldots, f((\tilde{x}_{k-1} + x_k))(t) - f(x_k(t))\big)$$

and matrix $\tilde{L}$ of appropriate dimension. Clearly systems (1) coupled via (2) (locally) synchronize if $\tilde{x}(t) \to 0$ as $t \to \infty$. In this section we present conditions for which the zero solution of (12) is (locally) uniformly asymptotically stable. Note that uniform asymptotic stability of the zero solution of the synchronization error system is in fact a stronger property than having asymptotic convergence of $\tilde{x}(t)$ to 0.

Because we are interested in *local* synchronization it makes sense to investigate the stability of the error system in the first order approximation. To simplify notation we denote by $J_f(t) = J_f(\xi(t))$ the Jacobian of the function $f : \mathbb{R}^n \to \mathbb{R}^n$ evaluated along $\xi(t)$, $\xi \in \mathcal{C}([t_0 - \tau, \infty), \mathbb{R}^n)$. We then have the following result that is illustrated in Fig. 2.

**Lemma 4.** *Consider the coupled system* (1), (2) *and suppose that there exists a constant $M > 0$ and a non-empty set $\mathcal{S}_M \subset \mathbb{R}_+ \times \mathbb{R}_+$ such that $\phi \in \mathcal{C}_M$ implies $x_t \in \mathcal{C}_M$ for $(\sigma, \tau) \in \mathcal{S}_M$. Let $\xi(t) = \frac{1}{k}\sum_{i=1}^{k} x_i(t)$ and $J_f(t) = J_f(\xi(t))$. Then if there is a non-empty set $\mathcal{S} \subset \mathbb{R}_+ \times \mathbb{R}_+$ such that for $(\sigma, \tau) \in \mathcal{S}$ the zero solution of the variational equation*

$$\dot{\eta}(t) = J_f(t)\eta(t) - \sigma BC\eta(t-\tau), \tag{13}$$

*is uniformly asymptotically stable, then the systems* (1) *with coupling* (2) *locally synchronize when*

$$(\sigma, \tau) \in \bigcap_{j=2}^{k} \mathcal{S}_j \cap \mathcal{S}_M \neq \emptyset$$

*with*

$$\mathcal{S}_j := \left\{(\sigma, \tau) | (\sigma\lambda_j/2, \tau) \in \mathcal{S}\right\}, \tag{14}$$

*where $\lambda_j$ ($j = 2, \ldots, k$) are the non-zero eigenvalues of $L$.*

**Proof.** Consider the change of variables $\zeta(t) = (U \otimes I_n)x(t)$ where $U$ is the non-singular matrix $U$ that satisfies $\|U \otimes I_n\| = 1$ and

$$ULU^{-1} = \begin{pmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \end{pmatrix} =: \Lambda.$$

The existence of such $U$ is guaranteed by symmetry of $L$. Note that the property that $\|U \otimes I_n\| = 1$ is equivalent to the property that the maximum norm (the maximal absolute row sum) of $U$ equals 1. Then, since the eigenvector corresponding to the zero eigenvalue of $L$ is in span $(\mathbf{1}_k)$, $\|U \otimes I_n\| = 1$ implies that the first row of $U$ has entries $\frac{1}{k}$, hence $\zeta_1(t) = \xi(t) = \frac{1}{k}\sum_{i=1}^{k} x_i(t)$. Moreover we have

$$U\mathbf{1}_k = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^{\top}$$

which implies that if $\zeta_j(t) \equiv 0$ for $j = 2, \ldots, k$, then $x_i(t) = \zeta_1(t) = \xi(t)$ for all $i$, i.e. the coupled system is synchronized. In the new variables $\zeta$, the coupled system (1), (2) satisfies

$$\dot{\zeta}(t) = (U \otimes I_n)F((U^{-1} \otimes I_n)\zeta(t)) - \sigma(\Lambda \otimes BC)\zeta(t-\tau). \tag{15}$$

Clearly $x_t \in \mathcal{C}_M$ for $\phi \in \mathcal{C}_M$ implies $\zeta_t \in \mathcal{C}_M$ for $\varphi \in \mathcal{C}_M$, $\varphi(\theta) = (U \otimes I_n)\phi(\theta)$, $-\tau \leq \theta \leq 0$. Linearization of (15) around col $(\xi_t, \mathbf{0}_t, \ldots, \mathbf{0}_t)$, where $\mathbf{0}_t$ denotes the zero solution in $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$, gives $k$ linear equations

$$\dot{\eta}_i(t) = J_f(t)\eta_i(t) - \sigma\lambda_i BC\eta_i(t-\tau), \quad i = 1, 2, \ldots, k. \tag{16}$$

The conditions of the lemma imply that the zero solution of the $j^{\text{th}}$ system (16), $j = 2, \ldots, k$, is uniformly asymptotically stable for $(\sigma, \tau) \in \mathcal{S}_j$. Then if $(\sigma, \tau) \in \cap_{j=2}^{k} \mathcal{S}_j$, with $\mathcal{S}_j$ as in (14), the zero solutions of all systems (16) for $j = 2, \ldots, k$ are uniformly asymptotically stable. Because the solutions $(\zeta_t, \eta_t)$ must satisfy the Eqs. (15), (16), cf. [47], and the change of variables is well-defined, we conclude that the coupled system (1), (2) locally synchronizes for $(\sigma, \tau) \in \cap_{j=2}^{k} \mathcal{S}_j \cap \mathcal{S}_M$.

**Remark 1.** Note that uniform asymptotic stability of the zero solution of systems (13) implies exponential stability of the zero solution, cf. [48].

**Remark 2.** Lemma 4 is basically a restatement of the *fundamental lemma* in [34] (see also [29] for the non-delayed case) with the exception that we require explicitly the solutions of the coupled system to be bounded.

A problem with Lemma 4 is that $\xi(t)$ is not known *a priori*, so the result is not directly useful in applications. Given that the solutions of the coupled system satisfy $x_t \in \mathcal{C}_M$, which implies that $|\xi(t)| \leq M$, one can apply the lemma if a Lyapunov functional can be constructed that has a negative definite derivative along solutions of (13) with $J_f(t) = J_f(\tilde{\xi}(t))$ for all functions $\tilde{\xi} \in \Xi_M$. Here $\Xi$ denotes the space of continuous functions bounded by $M$, $\left|\tilde{\xi}(t)\right| \leq M$ for all $t$, and clearly $\xi \in \Xi_M$. In Appendix A we construct such a (quadratic) Lyapunov functional.

However, in applications the solutions $x_t$ of the coupled system often do not explore the whole space $\mathcal{C}_M$. For instance, if $\mathcal{C}_M$

contains multiple attractors, the question of interest is under which conditions a local attractor $\mathcal{A}$ of the isolated system $\dot{s}(t) = f(s(t))$ is also the local attractor for the coupled system (1), (2). In such case the requirement of having the derivative of a Lyapunov functional negative definite for every $\tilde{\xi} \in \Xi_M$ is too restrictive, as we only need to consider only functions $\tilde{\xi}(t)$ near the attractor, or ultimately, solutions of $\dot{\tilde{\xi}}(t) = f(\tilde{\xi}(t))$ on the attractor.

In the remainder of this section we focus on the case where the isolated system

$$\dot{s}(t) = f(s(t))$$

has an attractor $\mathcal{A} \subset \mathbb{R}^n$. We assume that there exists a neighborhood $\mathcal{U}$ of $\mathcal{A}$ that is contained in its basin of attraction. (Note that in general the basin of attraction of $\mathcal{A}$ is not required to be, or contain, a neighborhood of $\mathcal{A}$, i.e. when $\mathcal{A}$ is an attractor in weak, Milnor, sense, cf. [49].) Let $\partial\mathcal{U}$ and $\overline{\mathcal{U}}$ be the boundary, respectively closure, of $\mathcal{U}$, and assume that $\mathcal{U}$ is inflowing with respect to $f$ [50,51]. That is, the vectorfield $f$ is pointing strictly inward on $\partial\mathcal{U}$, i.e., there is a positive constant $\epsilon$ such that

$$\langle N(s), f(s)\rangle \le -\epsilon < 0, \quad \forall s \in \partial\mathcal{U},$$

with $N(s)$ being the outward normal of $\mathcal{U}$ at $s$ and $\langle\cdot,\cdot\rangle$ the standard inner-product in $\mathbb{R}^n$. Let $\mathcal{C}_{\mathcal{U}} = \{\phi = \mathrm{col}\,(\phi_1, \ldots, \phi_k) \in \mathcal{C} \mid \phi_i(\theta) \in \mathcal{U}, -\tau \le \theta \le 0\}$.

**Theorem 5.** *Suppose that there is a non-empty set $\mathcal{S} \subset \mathbb{R}_+ \times \mathbb{R}_+$ such that for any $(\sigma,\tau) \in \mathcal{S}$ and any $\tilde{\xi} \in \mathcal{C}(\mathbb{R}, \overline{\mathcal{U}})$ the zero solution of the system*

$$\dot{\eta}(t) = J_f(t)\eta(t) - \sigma BC\eta(t-\tau),$$

*with $J_f(t) = J_f(\tilde{\xi}(t))$, is uniformly asymptotically stable. If $(\sigma,\tau) \in \bigcap_{j=2}^k \mathcal{S}_j$, with $\mathcal{S}_j$ defined in (14), then there is a $\delta = \delta(\sigma,\tau) > 0$ such that the solutions of the coupled system (1), (2) with initial data $\phi \in \mathcal{C}_{\mathcal{U}}$, $\|\phi_i - \phi_j\| < \delta$, are contained in $\mathcal{C}_{\mathcal{U}}$ and, moreover, the coupled system (1), (2) locally synchronizes.*

**Proof.** By assumption and Remark 1, the zero solution of each system

$$\dot{\eta}_j(t) = J_f(t)\eta_j(t) - \sigma\lambda_j BC\eta(t-\tau), \quad j = 2,\ldots,k, \tag{17}$$

is exponentially stable. Thus there exist positive constants $\alpha, \beta$ such that for $\psi \in \mathcal{C}([-\tau, 0], \mathbb{R}^{(k-1)n})$ we have

$$|\eta(t; t_0, \psi)| \le \beta e^{-\alpha(t-t_0)} \|\psi\|,$$

where $\eta(t) = \mathrm{col}\,(\eta_2(t), \ldots, \eta_k(t))$. Choose $K = \left(1 + \frac{1}{2\alpha}\right)\beta^2 e^{2\alpha\tau}$ and let $\delta$ be small enough to ensure that the linearization dominates the nonlinear terms and such that $\sigma K\delta\|BC\| < \epsilon$. Assume that there is a $t_1 > t_0$ such that $x_t \in \mathcal{C}_{\mathcal{U}}$ for $t_0 \le t < t_1$, and $x_i(t_1) \in \partial\mathcal{U}$ for some $i$. Without loss of generality we assume $i = 1$, i.e. $x_1(t_1) \in \partial\mathcal{U}$ and $x_j(t) \in \mathcal{U}$ for all $j = 2, \ldots, k$ for all $t \in [t_0, t_1]$. Define $\zeta_j(t) = x_1(t) - x_j(t), j = 2, \ldots, k$, which gives that

$$\begin{aligned} \dot{x}_1(t) &= f(x_1(t)) - \sigma(\ell^\top \otimes BC)\zeta(t-\tau), \\ \dot{\zeta}(t) &= \tilde{F}(x_1(t), \zeta(t)) - \sigma(\tilde{L} \otimes BC)\zeta(t-\tau), \end{aligned} \tag{18}$$

with

$$\begin{aligned} \zeta(t) &= \mathrm{col}\,(\zeta_2(t), \ldots, \zeta_k(t)) \\ \tilde{F}(x_1(t), \zeta(t)) \;&=\; \mathrm{col}\,(f(x_1(t)) - f(x_1(t) \\ &\quad - \zeta_2(t)), \ldots, f(x_1(t)) - f(x_1(t) - \zeta_k(t))) \end{aligned}$$

and

$$\begin{pmatrix} 0 & \ell^\top \\ \mathbf{0}_{k-1} & \tilde{L} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}_{k-1}^\top \\ \mathbf{1}_{k-1} & -I_{k-1} \end{pmatrix} L \begin{pmatrix} 1 & \mathbf{0}_{k-1}^\top \\ \mathbf{1}_{k-1} & -I_{k-1} \end{pmatrix}^{-1}.$$

Note that the eigenvalues of $\tilde{L}$ are the non-zero eigenvalues of $L$ and $|\ell| \le 1$ since $\sum_{j\in\mathcal{N}_i} a_{ij} \le 1$. Using similar arguments as in the proof of Lemma 4 we conclude that the linearized $\zeta$-dynamics are exponentially contracting on the interval $[t_0, t_1)$ and the following estimate holds:

$$|\zeta(t)| \le K\exp(-\gamma(t-t_0))\delta \le K\delta,$$

with positive constant $\gamma$. See [48], Theorems 4.5 and 4.6. The assumption of $\overline{\mathcal{U}}$ being inflowing invariant implies that $\mathcal{U}$ is positively invariant with respect to the dynamics

$$\dot{s}(t) = f(s(t)) + p(t),$$

with "perturbation" $p \in \mathcal{C}(\mathbb{R}, \mathbb{R}^n)$, $\sup_{t\in\mathbb{R}} |p(t)| < \epsilon$. Hence for $\delta < \frac{\epsilon}{\sigma K\|BC\|}$ we have $t_1 = \infty$, i.e. all solutions $x_t \in \mathcal{C}_{\mathcal{U}}$. Since we assume the zero solution of (17) to be uniformly asymptotically stable for every $\tilde{\xi} \in \Xi$, and clearly $\xi(t) = \frac{1}{k}\sum_{i=1}^k x_i(t)$ is in $\Xi$, we can apply Lemma 4 and conclude that the systems locally synchronize.

**Theorem 6.** *Suppose that $\mathcal{A}$ is an asymptotically stable equilibrium point or an asymptotically stable periodic orbit and let $\tilde{\xi}(t)$ be a solution of the isolated system $\dot{\tilde{\xi}}(t) = f(\tilde{\xi}(t))$, $\tilde{\xi}(t_0) \in \mathcal{A}$, i.e. $\tilde{\xi}\cdot$ is a solution of the isolated system on $\mathcal{A}$. Assume that there is a non-empty set $\mathcal{S} \subset \mathbb{R}_+ \times \mathbb{R}_+$ such that for any $(\sigma,\tau) \in \mathcal{S}$ the zero solution of the system*

$$\dot{\eta}(t) = J_f(t)\eta(t) - \sigma BC\eta(t-\tau),$$

*with $J_f(t) = J_f(\tilde{\xi}(t))$, is uniformly asymptotically stable. Then if $(\sigma,\tau) \in \bigcap_{j=2}^k \mathcal{S}_j$ the conclusions of Theorem 5 hold.*

**Proof.** If $\mathcal{A}$ is an equilibrium point, it has to be an asymptotically stable equilibrium of the isolated system, and linearization around that equilibrium shows that it is an exponentially stable equilibrium for the whole network. If $\mathcal{A}$ is a periodic orbit, then is it an orbitally asymptotically (and exponentially) stable periodic orbit for the isolated system. Let $\Xi_{\mathcal{A}}$ be the set of all solutions of $\dot{\xi}(t) = f(\xi(t))$ on $\mathcal{A}$. Note that the synchronized solution of the coupled system is a solution $\mathrm{col}\left(\tilde{\xi}(\cdot), \ldots, \tilde{\xi}(\cdot)\right)$ with $\tilde{\xi}(\cdot) \in \Xi_{\mathcal{A}}$. Consider the system (18) and note that the solution $\mathrm{col}\left(\tilde{\xi}(\cdot), 0, \ldots, 0\right)$ of this system corresponds to the synchronized solution of the coupled system. Linearize around $\mathrm{col}\left(\tilde{\xi}(\cdot), 0, \ldots, 0\right)$ to obtain

$$\dot{\zeta}(t) = (I_{k-1} \otimes J_f(\tilde{\xi}(t)))\zeta(t) - \sigma\left(\begin{pmatrix} 0 & \ell^\top \\ \mathbf{0}_{k-1} & \tilde{L} \end{pmatrix} \otimes BC\right)\zeta(t-\tau).$$

Since $J_f(\tilde{\xi}(t))$ is periodic we can use Floquet theory, cf. [38]. First we find a set of new variables in which the equation above has the same structure except that the matrix $\tilde{L}$ is diagonal. (See the proof of Theorem 5). Note the system now has a block triangular structure, hence the monodromy matrix of the system has triangular structure too. Then the assumptions on the theorem imply that all Floquet multipliers except one are contained in the (open) unit disc in the complex plane $\mathbb{C}$, hence the coupled system locally synchronizes. Moreover, since the Floquet multipliers do not depend on the starting time, cf. [38] Section 8.1 Lemma 1.3, it is sufficient to linearize around one solution $\tilde{\xi}(t)$ on $\mathcal{A}$.

**Example 4** (*Local Synchronization of FitzHugh–Nagumo Model Neurons*)**.** We will determine a set $\mathcal{U}$ for the FitzHugh–Nagumo model neuron (7) and subsequently we apply Theorem 5 and construct the local synchronization diagram $\mathcal{S}$. Consider an isolated system

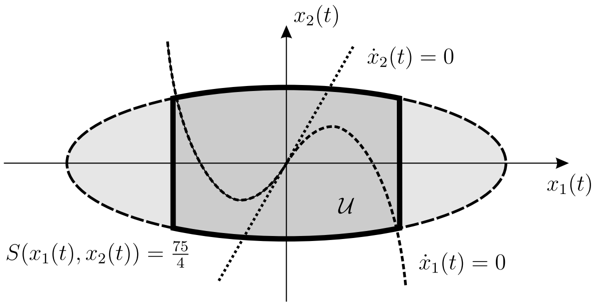


**Fig. 3.** Construction of the set $\mathcal{U}$.

(7) ($u_i(t) \equiv 0$) and the storage function (8). Then

$$\dot{S}(x_i(t)) = -x_{i,1}^2(t)\left(\tfrac{1}{3}x_{i,1}^2(t) - 1\right) - \tfrac{8}{10}x_{i,2}^2(t),$$

and we see that $\dot{S}(x_i(t)) < 0$ on $\mathbb{R}^2 \setminus \{x_i(t) \in \mathbb{R}^2 | x_{i,2}(t) = 0 \text{ and } x_{i,1}^2(t) \leq 3\}$. This means that the dynamics (8) is inflowing with respect to the set $\mathcal{U}^* := \left\{x_i(t) \in \mathbb{R}^n | S(x_i(t)) < \frac{75}{4}\right\}$. We may choose $\mathcal{U} = \mathcal{U}^*$ but we can easily determine a set contained in $\mathcal{U}^*$ such that (8) is inflowing with respect to that set. To this end we determine the nullcline $0 = \dot{x}_{i,1}(t) = x_{i,1}(t) - \frac{1}{3}x_{i,1}^3(t) - x_{i,2}(t)$ and we compute the intersection of the nullcline with the curve $S(x_i(t)) = \frac{75}{4}$. The points of intersection are (approximately) $(-2.2652, 1.6092)$ and $(2.2652, -1.6092)$. Since $\dot{x}_{i,1}(t) > 0$ ($\dot{x}_{i,1}(t) < 0$) left (right) of the nullcline, we conclude that the set

$$\mathcal{U} = \left\{x_i(t) \in \mathcal{U}^* | \left|x_{i,1}(t)\right| < 2.3\right\}$$

is such that (8) is inflowing with respect to $\mathcal{U}$. See Fig. 3 for a graphical representation of the construction of $\mathcal{U}$.

Given the set $\mathcal{U}$ we can determine the set $\mathcal{S}$. We will do so with the help of the machinery presented in Appendix A. For the FitzHugh–Nagumo model neuron (7) we find

$$J_f(\xi(t)) = \begin{pmatrix} 1 - \tilde{\xi}_1^2(t) & -1 \\ \frac{8}{100} & -\frac{64}{1000} \end{pmatrix},$$

where $\tilde{\xi}_1(t)$ the first component of $\tilde{\xi}(t)$. Since $\tilde{\xi} : \mathbb{R} \to \overline{\mathcal{U}}$ we have $\left|\tilde{\xi}_1(t)\right| \leq 2.3$ for all $t$. Then it is easy to see that, for each $t$,

$$J_f(\tilde{\xi}(t)) = \sum_{j=1}^{2} \nu_j(t)\bar{J}_{f,j}, \quad \sum_{j=1}^{2} \nu_j(t) = 1,$$

with

$$\bar{J}_{f,1} = \begin{pmatrix} 1 - 2.3^2 & -1 \\ \frac{8}{100} & -\frac{64}{1000} \end{pmatrix} \quad \text{and} \quad \bar{J}_{f,2} = \begin{pmatrix} 1 & -1 \\ \frac{8}{100} & -\frac{64}{1000} \end{pmatrix}.$$

By solving the LMIs (Linear Matrix Inequalities) (A.2) with $\bar{G}_j = \bar{J}_{f,j}$, $j = 1, 2$, and

$$G_2 = -\sigma BC = -\sigma \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

the Matlab LMI solver allows us to construct the synchronization diagram $\mathcal{S}$ depicted in Fig. 4.

Next we determine the set of values of $\sigma$ and $\tau$ for which we get local synchronization in a network with $k = 16$ FitzHugh–Nagumo model neurons with coupling (2) and network structure shown in Fig. 5.

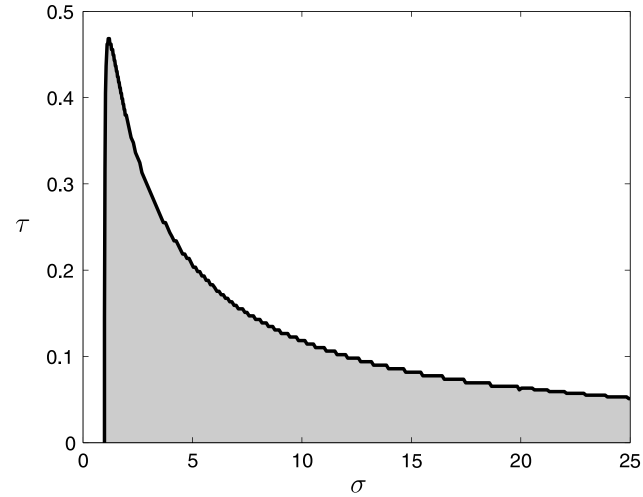


**Fig. 4.** The local synchronization diagram $\mathcal{S}$ for the FitzHugh–Nagumo model neuron.

The non-zero eigenvalues of the corresponding Laplacian matrix $L$ are approximately

$\lambda_2 = 0.1799$, $\lambda_3 = 0.2110$, $\lambda_4 = 0.4165$,
$\lambda_5 = 0.5830$, $\lambda_6 = 0.6132$, $\lambda_7 = 0.7031$,
$\lambda_8 = 0.8503$, $\lambda_9 = 0.9036$, $\lambda_{10} = 0.9702$,
$\lambda_{11} = 1.0615$, $\lambda_{12} = 1.0756$, $\lambda_{13} = 1.1362$,
$\lambda_{14} = 1.2865$, $\lambda_{15} = 1.3697$, $\lambda_{16} = 1.4397$.

Fig. 5 shows the scaled copies $\mathcal{S}_j$ (light shades) of the synchronization diagram $\mathcal{S}$ and the intersection (dark shade). Theorem 5 tells that the network with $k = 16$ FitzHugh–Nagumo neuron (7) with coupling (2) and network structure shown in Fig. 5 locally synchronizes if $\sigma$ and $\tau$ belong to this dark shaded set. $\triangle$

As the example above illustrates, if the synchronization diagram has a specific shape, then $\bigcap_{j=2}^{k} \mathcal{S}_j = \mathcal{S}_2 \cap \mathcal{S}_k$.

**Lemma 7.** *Let $\mathcal{S} \neq \emptyset$ be given and let*

$$\mathcal{T}_{\mathcal{S}} = \{\tau | \exists \sigma \text{ such that } (\sigma, \tau) \in \mathcal{S}\}.$$

*If for each fixed $\tau^* \in \mathcal{T}_{\mathcal{S}}$ the set $\{\sigma | (\sigma, \tau^*) \in \mathcal{S}\}$ is connected, then $\bigcap_{j=2}^{k} \mathcal{S}_j = \mathcal{S}_2 \cap \mathcal{S}_k$.*

**Proof.** Straightforward.

Since the FitzHugh–Nagumo model neuron dynamics (7) is two-dimensional, $\mathcal{U}$ is positively invariant for the isolated system and contains an unstable equilibrium point, there must exist a periodic orbit in $\mathcal{U}$. (This follows from the Poincaré–Bendixson Theorem, cf. [52].) In fact, system (7) is a Liénard system

$$\dot{u}(t) = v(t)$$
$$\dot{v}(t) = -G(u(t))v(t) - F(u(t))$$

with $u(t) = x_{i,1}(t)$, $v(t) = x_{i,1}(t) - \frac{1}{3}x_{i,1}^3(t) - x_{i,2}(t)$, $G(u(t)) = u^2(t) - 1 - \frac{64}{1000}$ and $F(u(t)) = \frac{8}{100}(1 - \frac{8}{10})u(t) + \frac{64}{3000}u^3(t)$. Liénard's theorem, cf. [52], implies that the periodic orbit of the FitzHugh–Nagumo model neuron (7) in $\mathcal{U}$ is asymptotically stable and unique. Thus the system has a periodic attractor $\mathcal{A}$ in $\mathcal{U}$ which means that we could have applied Theorem 6 in the example. Fig. 6 shows the local synchronization diagram $\mathcal{S}$ which is determined by Theorem 6.

This diagram is obtained by computing the Floquet multipliers of system

$$\dot{\eta}(t) = \begin{pmatrix} 1 - \tilde{\xi}_1^2(t) & -1 \\ \frac{8}{100} & -\frac{64}{1000} \end{pmatrix} \eta(t) - \sigma \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \eta(t - \tau)$$

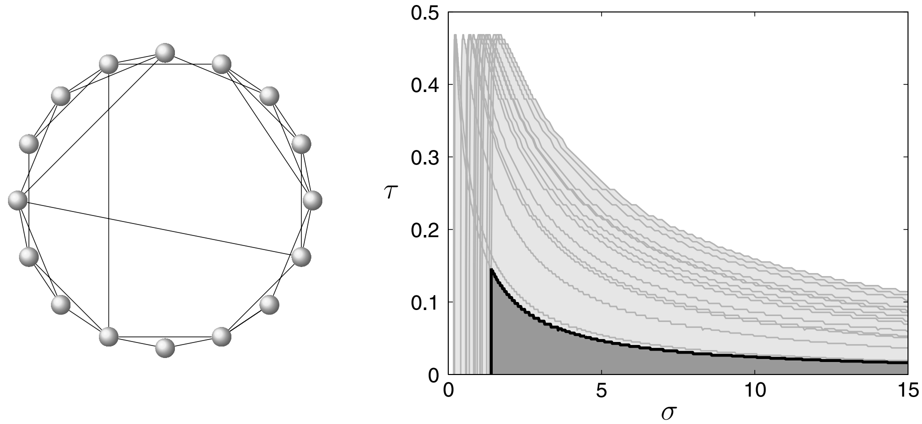


**Fig. 5.** Left: The network of Example 4. The interconnection weights are all the same, $a_{ij} = \frac{1}{5}$. Right: The set of values of $\sigma$ and $\tau$ (dark shade) for which the network locally synchronizes.

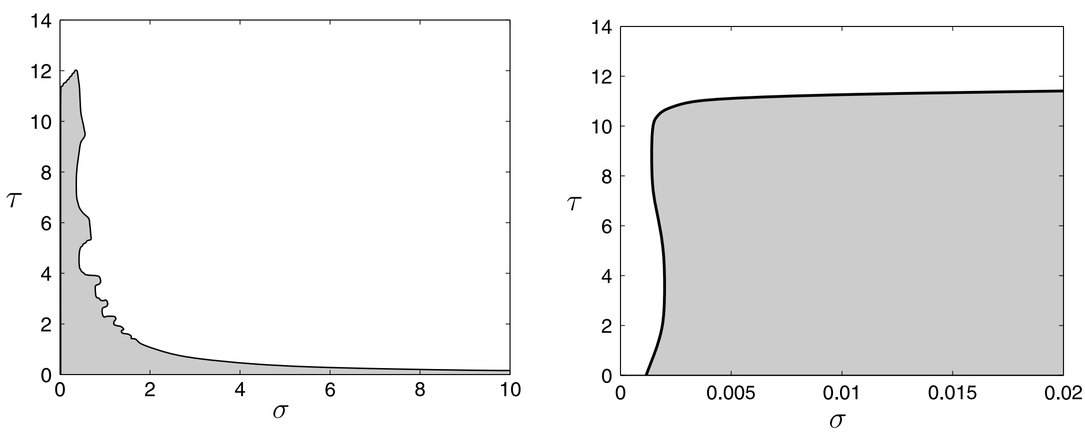


**Fig. 6.** Left: the local synchronization diagram $\mathcal{S}$ for the FitzHugh–Nagumo model neuron of Example 4 estimated using Theorem 6. Right: the set $\mathcal{S}$ for small $\sigma$.

where $\tilde{\xi}_1(t)$ satisfies

$$\dot{\tilde{\xi}}_1(t) = \tilde{\xi}_1(t) - \tfrac{1}{3}\tilde{\xi}_1^3(t) - \tilde{\xi}_2(t)$$
$$\dot{\tilde{\xi}}_2(t) = \tfrac{8}{100}\left(\tilde{\xi}_1(t) - \tfrac{8}{10}\tilde{\xi}_2(t)\right)$$

with $\tilde{\xi}(-\tau) \in \mathcal{A}$. These computations are done with DDE-BIFTOOL [53]. We have observed that this synchronization diagram practically coincides with the one shown in Fig. 4 for $\sigma \geq 10$. This may be explained by the fact that for large $\sigma$ the non-linear terms in $f$ are suppressed, hence the conservatism of the LMI based approach of Example 4 is reduced.

## 4. Conditions for global synchronization

We differentiate the outputs $y_i(t)$ to obtain

$$\dot{y}_i(t) = Cf(x_i(t)) + CBu_i(t).$$

Since $CB$ is assumed to be (similar to) a positive definite matrix we can find $n - m$ coordinates $z_i(t)$ complementary to $y_i(t)$ and write (1) as

$$\begin{cases} \dot{z}_i(t) = q(z_i(t), y_i(t)), \\ \dot{y}_i(t) = a(z_i(t), y_i(t)) + CBu_i(t), \end{cases} \tag{19}$$

where $z_i(t) \in \mathbb{R}^{n-m}$ and the functions $q : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}$, $a : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$ are sufficiently smooth. See [54] for details regarding this transformation. Obviously, if $n = m$ the $z_i$-dynamics are absent. For convenience we again set $CB = I_m$. The transformation of (1) into (19) allows to write the coupled system as

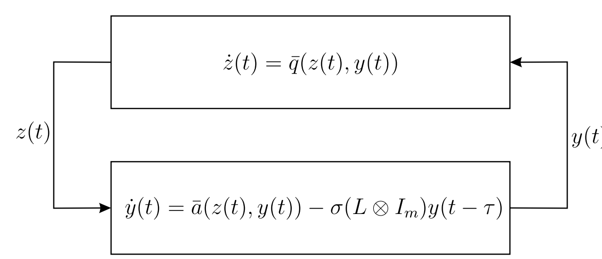


**Fig. 7.** The structure of the coupled system in new coordinates.

$$\begin{cases} \dot{z}(t) = \bar{q}(z(t), y(t)), \\ \dot{y}(t) = \bar{a}(z(t), y(t)) - \sigma(L \otimes I_m)y(t-\tau), \end{cases} \tag{20}$$

with

$$z(t) = \mathrm{col}\,(z_1(t), \ldots, z_k(t)),$$
$$y(t) = \mathrm{col}\,(y_1(t), \ldots, y_k(t)),$$
$$\bar{q}(z(t), y(t)) = \mathrm{col}\,(q(z_1(t), y_1(t)), \ldots, q(z_k(t), y_k(t))),$$
$$\bar{a}(z(t), y(t)) = \mathrm{col}\,(a(z_1(t), y_1(t)), \ldots, a(z_k(t), y_k(t))).$$

The system (20) consists of a system of ordinary differential equations (the $z$-dynamics) coupled with a system of delay differential equations (the $y$-dynamics). See also Fig. 7.

Note that the coupling strength $\sigma$ and the time-delay $\tau$ appear only in the $y$-system. For large $\sigma$ the term $-\sigma(L \otimes I_m)y(t-\tau)$

is the dominant part of the $y$-dynamics, which suggests that, if $z(t)$ and $y(t)$ are bounded, $\tau$ is sufficiently small and in the limit $\sigma \to \infty$, we may have synchronization of the outputs of the systems, i.e. $y_i(t) \to y_j(t)$ as $t \to \infty$ for all $i, j$. Then to have synchronization in the sense of Definition 2 we need an additional condition on the $z$-dynamics that ensures that $z_i(t) \to z_j(t)$ as $t \to \infty$ for all $i, j$. Clearly, assuming $q(z_i(t), y_i(t))$ to be linear in both $z_i(t)$ and $y_i(t)$,

$$q(z_i(t), y_i(t)) = Q_1 z_i(t) + Q_2 y_i(t)$$

$Q_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ a Hurwitz matrix and $Q_2 \in \mathbb{R}^{(n-m)\times m}$, is sufficient to get the desired behavior. However, the assumption of linearity of $q$ is obviously quite restrictive. Fortunately there is a class of non-linear systems (of ordinary differential equations) that have the property that the states of the systems asymptotically match when they are driven by a common signal. These systems are known as *convergent systems*, cf. [55–57]. We refer to these references for the precise definition of convergence and the many more interesting properties of convergent systems. For our purpose we only need a (sufficient) condition for the system $\dot{z}_i(t) = q(z_i(t), y_i(t))$ to be (exponentially) convergent with respect to input $y_i(t)$.

**Lemma 8** (*[55]*). *If there exists a positive definite matrix $P \in \mathbb{R}^{(n-m)\times(n-m)}$ such that all eigenvalues of the symmetric matrix*

$$\frac{1}{2}\left(P\left(\frac{\partial q}{\partial z_i}(z_i, y_i)\right) + \left(\frac{\partial q}{\partial z_i}(z_i, y_i)\right)^{\top} P\right)$$

*are negative and bounded away from zero for all $z_i \in \mathbb{R}^m$, $y_i \in \mathcal{Y}$, where $\mathcal{Y} \subset \mathbb{R}^m$ is compact, then the system*

$$\dot{z}_i(t) = q(z_i(t), y_i(t))$$

*is exponentially convergent with piecewise continuous inputs $y_i : \mathbb{R} \to \mathcal{Y}$.*

It was shown in [33] that, if the sub-system $\dot{z}_i(t) = q(z_i(t), y_i(t))$ satisfies the conditions of Lemma 8 and if the solutions of the coupled system (19), (2) are bounded for $(\sigma, \tau) \in \mathcal{S}_M$, with $\mathcal{S}_M$ defined in Lemma 4, then there exist positive constants $\bar{\sigma}$, $\bar{\rho}$ such that if $(\sigma, \tau) \in \mathcal{S}^* \cap \mathcal{S}_M$, then the coupled system (19), (2) (and thus the coupled system (1), (2)) globally synchronizes, where the set $\mathcal{S}^*$ is defined by

$$\mathcal{S}^* = \{(\sigma, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+ | \bar{\sigma} \le \sigma \text{ and } \sigma\tau \le \bar{\rho} < \tfrac{1}{2}\}.$$

**Remark 3.** The result of [33] actually holds for a more general class of networks; the only assumption on the network is that the associated graph $\mathcal{G}$ is strongly connected.

Looking at the shape of the set $\mathcal{S}^*$, see Fig. 8, and having Lemma 7 in mind, we may expect local synchronization in any network for $(\sigma, \tau)$ belonging to the intersection of $\mathcal{S}_M$ and two scaled copies of $\mathcal{S}^*$, where one copy is scaled over the $\sigma$-axis by a constant proportional to $\lambda_2$ and the scaling of the other copy is proportional to $\lambda_k$.

In fact, we can show that under these conditions the synchronization is even global. Let us first formally state the result of [33] for $k = 2$ coupled systems.

**Theorem 9.** *Consider $k = 2$ systems (19) with coupling (2). Assume that*

- *for $(\sigma, \tau) \in \mathcal{S}_M \neq \emptyset$ the solutions of the coupled system (19), (2) are bounded in the sense that there is a constant $M$ such that $x_t(t_0, \phi) \in \mathcal{C}_M$ for all $t \ge t_0$;*

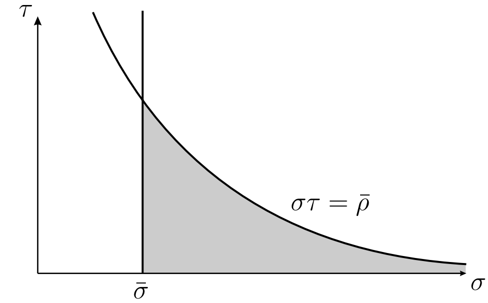


**Fig. 8.** The set $\mathcal{S}^*$.

- *the function $q(z_i(t), y_i(t))$ satisfies the condition of Lemma 8, i.e. the system $\dot{z}_i(t) = q(z_i(t), y_i(t))$ is exponentially convergent with input $y_i(t)$.*

*Then there exist positive constants $\bar{\sigma}$ and $\bar{\rho} < \frac{1}{2}$ such that if $(\sigma, \tau) \in \mathcal{S}^* \cap \mathcal{S}_M$,*

$$\mathcal{S}^* = \{(\sigma, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+ | \bar{\sigma} \le \sigma \text{ and } \sigma\tau \le \bar{\rho} < \tfrac{1}{2}\},$$

*$\mathcal{S}^* \cap \mathcal{S}_M \neq \emptyset$, then the coupled system (19), (2) globally synchronizes.*

We now extend this result to $k > 2$ systems as follows.

**Theorem 10.** *Consider $k > 2$ systems (19) with coupling (2) and let the assumptions of Theorem 9 be satisfied. Then the coupled system (19), (2) globally synchronizes if $(\sigma, \tau) \in \mathcal{S}_2^* \cap \mathcal{S}_k^* \cap \mathcal{S}_M \neq \emptyset$ with*

$$\mathcal{S}_j^* = \{(\sigma, \tau) | (\tfrac{1}{2}\lambda_j\sigma, \tau) \in \mathcal{S}^*\}.$$

The proof of Theorem 10 can be found in Appendix B.

**Example 5** (*Global Synchronization of two FitzHugh–Nagumo Neurons*)**.** We consider, again, the FitzHugh–Nagumo model neuron (7), with $z_i(t) = x_{2,i}(t)$ and $y_i(t) = x_{1,i}(t)$,

$$\begin{pmatrix} a(z_i(t), y_i(t)) \\ q(z_i(t), y_i(t)) \end{pmatrix} = \begin{pmatrix} y_i(t) - \frac{1}{3}y_i^3(t) - z_i(t) \\ \frac{8}{100}\left(y_i(t) - \frac{8}{10}z_i(t)\right) \end{pmatrix} = f(x_i(t)). \tag{21}$$

Clearly the condition of Lemma 8 is satisfied with $P = 1$. By Theorem 9 we know that there exist values for $\sigma$ and $\tau$ for which a network with FitzHugh–Nagumo model neuron synchronizes. We consider the case $k = 2$, hence have $\lambda_2 = \lambda_k = 2$. Let us determine the constants $c_1, c_2, c_3, c_4, c_5$ specified in the proof. Clearly we have $c_1 = \frac{-64}{1000}$ and $c_2 = \frac{8}{100}$. We set $U = 1$ and find $c_3 = 1$ and $c_4 = 1$. These estimated values follow readily since the only non-linearity is the cubic term in $a$ and $(y_1 - y_2)(y_1 - y_2 - \frac{1}{3}(y_1^3 - y_2^3)) \le (y_1 - y_2)^2$ for all $y_1, y_2 \in \mathbb{R}$. From Examples 2 and 3 we know the bound $B_2 = 30$ and we find that $\left|y_1 - y_2 - \frac{1}{3}(y_1^3 - y_2^3)\right| \le 1201\,|y_1 - y_2|$ for all $-30 \le y_1, y_2 \le 30$. Thus $c_5 = 1201$. We set $\bar{\rho} = 0.001$ (as in Example 3) and find that for $\sigma \ge 2.9$ there is a $\gamma > 1$ for which the matrix $W > 0$. Hence $\mathcal{S}^* = \{(\sigma, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+ | \sigma \ge 2.9 \text{ and } \sigma\tau \le 0.001\}$. △

The conditions for global synchronization in this example looks quite conservative. The main source of the possible conservatism is, as already remarked in Example 3, the cubic term in the function $a$ for the FitzHugh–Nagumo model neuron, which results in a large number for the constant $c_5$. On the other hand we would like to remark that conditions for global synchronization may look more conservative than they actually are. For instance, it is shown in [58] that two Lorenz systems (6) with non-delayed coupling (9) globally synchronize *if and only if* $\sigma \ge \bar{\sigma} = 135$. This value of $\bar{\sigma}$ is much larger than estimates obtained from computer simulations (with random initial conditions) which are typically about 2.5 [59].

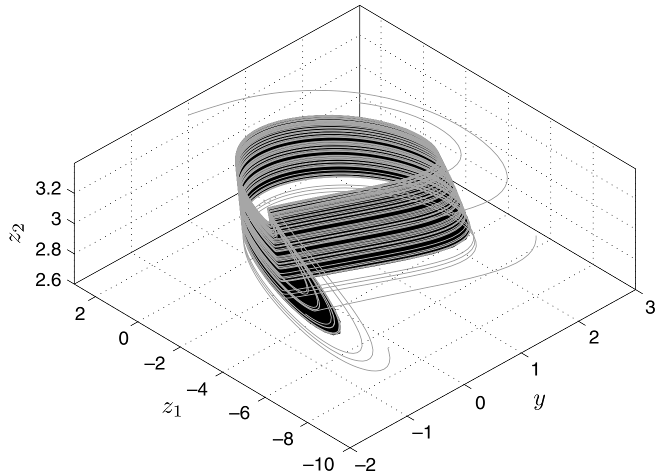


**Fig. 9.** The Hindmarsh–Rose chaotic attractor.

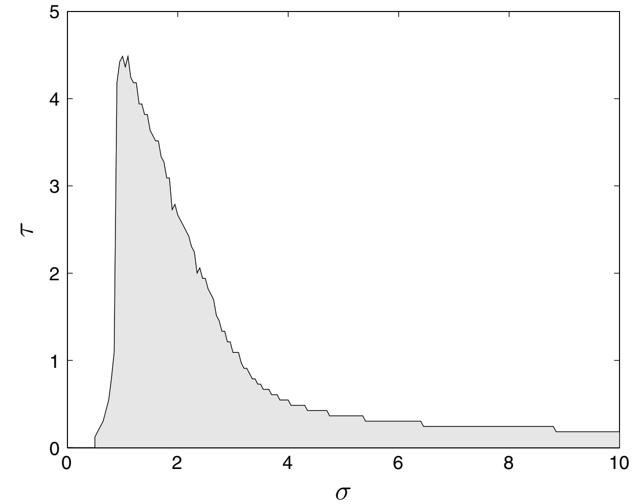


**Fig. 10.** Simulated stability region for two coupled Hindmarsh–Rose neurons.

## 5. A simulation example

In this section we will illustrate by means of an simulation example that the results developed above can also be applied to more complex systems. Consider a network of Hindmarsh–Rose neurons [60]

$$\begin{cases}\dot{z}_{1,i}(t) = 1 - 5y_i^2(t) - z_{1,i}(t),\\ \dot{z}_{2,i}(t) = 0.005(4(y_i(t) + 1.6180) - z_{2,i}(t)),\\ \dot{y}_i(t) = -y_i^3(t) + 3y_i^2(t) + z_{1,i}(t) - z_{2,i}(t) + 3.25 + u_i(t),\end{cases} \tag{22}$$

that interact via coupling (2). It is well known that the Hindmarsh–Rose neuron has a chaotic attractor, part of which is depicted in Fig. 9.

As proven in [61,40], the Hindmarsh–Rose neuron is strictly semi-passive with the quadratic storage function

$$S(y_i, z_{1,i}, z_{2,i}) = \tfrac{1}{2}y_i^2 + \gamma z_{1,i}^2 + 25z_{2,i}^2, \tag{23}$$

with $0 < \gamma < \frac{4\eta_1(1-\eta_2)}{25}$, $\eta_1, \eta_2 \in (0, 1)$, and function

$$\begin{aligned}H(y_i, z_{1,i}, z_{2,i}) = {} & \eta_1 y_i^4 - 3y_i^3 - \tfrac{1}{4\gamma(1-\eta_2)}y_i^2 - 3.25y_i\\ & + \left(\gamma\eta_2 - \tfrac{\gamma^2 5^2}{4(1-\eta_1)}\right) z_{1,i}^2 - \gamma z_{1,i} + \tfrac{1}{4}z_{2,i}^2\\ & - 1.6180z_{2,i} + \gamma(1-\eta_2)\left(z_{1,i} - \tfrac{1}{2\gamma(1-\eta_2)}y_i\right)^2\\ & + (1-\eta_1)\left(y_i^2 + \tfrac{5\gamma}{2(1-\eta_1)}z_{1,i}\right)^2.\end{aligned} \tag{24}$$

Thus Corollary 2 implies that the solutions of the closed-loop system (22), (2) are uniformly bounded and ultimately bounded. In what follows we investigate by means of simulations for which values of the coupling strength $\sigma$ and the time-delay $\tau$ two coupled Hindmarsh–Rose neurons synchronize. Then this knowledge is used to determine the values of $\sigma$ and $\tau$ for which four coupled Hindmarsh–Rose neurons in a ring show synchronous behavior.

Thus we first focus on $k = 2$ Hindmarsh–Rose neurons which are coupled via

$$u_1(t) = \sigma(y_2(t-\tau) - y_1(t-\tau)), \tag{25a}$$

$$u_2(t) = \sigma(y_1(t-\tau) - y_2(t-\tau)). \tag{25b}$$

Note that the Hindmarsh–Rose equations are in the normal form (20) and that the internal dynamics of the Hindmarsh–Rose neuron, i.e. the $(z_1, z_2)$-dynamics, satisfy the condition of Lemma 8 with $P = I$ and

$$Q = -\begin{pmatrix}1 & 0\\ 0 & 0.005\end{pmatrix}. \tag{26}$$

Hence, by Theorem 9, two Hindmarsh–Rose neurons synchronize whenever $(\sigma, \tau) \in \bar{\mathcal{S}}^*$, with $\bar{\mathcal{S}}^*$ as in Theorem 9. This also implies the existence of $\mathcal{S} \supseteq \bar{\mathcal{S}}^*$ as in Theorem 5 such that the two Hindmarsh–Rose neurons locally synchronize if $(\sigma, \tau) \in \mathcal{S}$.

We have estimated, using numerical simulations with Matlab®, for which values of $\sigma$ and $\tau$ two coupled Hindmarsh–Rose neurons (22), (25) locally synchronize. The result is shown in Fig. 10.

We next consider a network consisting of four Hindmarsh–Rose neurons (22) which are coupled in a ring, see Fig. 11. The corresponding Laplacian matrix is

$$L = \frac{1}{2}\begin{pmatrix}2 & -1 & 0 & -1\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ -1 & 0 & -1 & 2\end{pmatrix}, \tag{27}$$

which has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1$ and $\lambda_4 = 2$. Since the synchronization diagram satisfies the conditions of Lemma 7, it follows from this lemma that the stability region for the four coupled Hindmarsh–Rose neurons can be obtained by taking the intersection of the stability region shown in Fig. 10 and a copy of this region scaled by a factor $\frac{1}{2}$ over the $\sigma$-axis. The estimated stability region for four Hindmarsh–Rose neurons in a ring is shown in Fig. 11(b) in the dark gray shade. The black thick line in this figure indicates the boundary of the stability region for the four Hindmarsh–Rose neurons obtained by simulations.

## 6. Discussion

We have considered undirected networks of systems that interact via a type of time-delayed diffusive coupling in which the delay appears in the whole coupling term. First, under the assumption that the systems are strictly semi-passive, the solutions of the coupled systems are shown to be bounded. Boundedness of solutions of the coupled system is important in the study of synchronization as it is an asymptotic notion, hence solutions have to be well-defined on the whole interval $[t_0 - \tau, \infty)$. Next we have derived sufficient conditions for local and global synchronization. In case of local synchronization, i.e., the asymptotic match of trajectories of the systems with initial data mutually sufficiently close, we have focused on the situation in which the isolated system has an attractor with an open neighborhood with inflowing boundary. Given the existence of a non-empty set $\mathcal{S}$ representing the local synchronization diagram, we have shown that if the coupling strength $\sigma$ and time-delay $\tau$ are taken from the intersection of scaled copies of $\mathcal{S}$, then the solutions

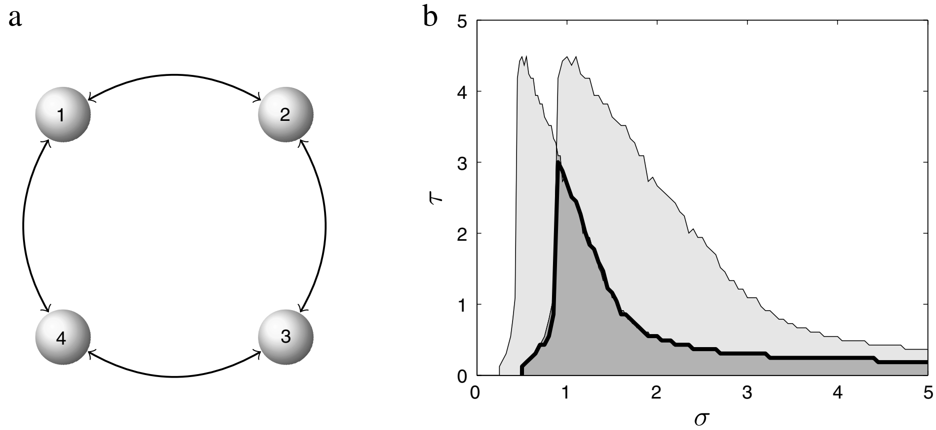


**Fig. 11.** (a) The ring network. (b) The stability region for two systems and its scaled copy (both in the light gray shade), the estimated stability region for the four coupled neurons in a ring (dark gray shade) and the boundary of the synchronization diagram for the four neurons obtained via numerical simulations (black thick line).

of the coupled system with initial data of the systems mutually close are contained in this neighborhood of the attractor and, in particular, the coupled system locally synchronizes. Next we have introduced a change of coordinates which allowed us to write the coupled system as a set of delay differential equations representing the output dynamics and a set of ordinary differential equations. This set of ordinary differential equations is independent of the time-delay, the coupling strength and the network topology and is assumed to satisfy a particular stability property (that depends only on the vectorfield $q$). We have shown that in that case there exists a non-empty set $\mathcal{S}^*$ which provides the values of $\sigma$ and $\tau$ for which $k = 2$ coupled systems with bounded solutions *globally* synchronize. Moreover, taking the intersection of two scaled copies of $\mathcal{S}^*$, with scaling factors being the smallest and largest eigenvalue of the Laplacian matrix, gives the values of $\sigma$ and $\tau^*$ for which a network with $k > 2$ systems globally synchronizes. It needs to be noted that the conditions for global synchronization, as well as the condition for delay-dependent boundedness of solutions of the coupled system, tend to be conservative. However, our results specify a class of systems for which there will exist values for the coupling strength and time-delay for which we have bounded solutions, global synchronization, and a global scaling law.

As already mentioned in the introduction, the conditions we have presented for local synchronization might remind one of the *Master Stability Function* approach, which was introduced in [28]. However, our assumptions are more restrictive. First of all we assume that the isolated system has an asymptotically stable attractor. More importantly, we also assume that the linear variational system (associated with transverse stability of the synchronization manifold) is uniformly stable along all possible functions on a bounded subset of the synchronization manifold. Although our conditions are obviously more restrictive than the ones in [28], they ensure that phenomena such as *bubbling* [31], a temporal loss of synchronization, cannot occur. Also, in case the attractor is an equilibrium point or periodic orbit, our conditions are not restrictive at all.

For zero time-delay, conditions that relate global synchronization of two coupled systems and global synchronization in a larger network are presented in [62–64]. The result of [62], widely known as the *Wu–Chua conjecture*, states that the condition for synchronization in two networks with Laplacian matrices $L_1$ and $L_2$ is that $\lambda_2(\sigma L_1) = \lambda_2(\sigma L_2)$. (Here $\lambda_2(\sigma L_j)$ denotes the smallest non-zero eigenvalue of the matrix $\sigma L_j$.) The conjecture was shown to be wrong in [65]. In particular, the Wu–Chua conjecture fails if the coupled systems lose synchrony when the coupling strength $\sigma$ is increased. However, for the class of systems we considered in Section 4 the conjecture *is* true [59]. Note that for this class of systems synchronization will be maintained for increasing coupling strength. The assumption that systems cannot lose synchrony for increasing coupling strength also plays a central role in the *Connection Graph Stability* method introduced in [63] for symmetric coupling and [64] for asymmetric coupling. A nice aspect of this approach is that it does not rely on the computation of the eigenvalues of the Laplacian matrix; instead the conditions follow from the characteristic path length of the underlying graph. For non-zero delay we cannot guarantee that the systems remain synchronized as the coupling strength is increased. In fact, our conditions show that, like in the Wu–Chua conjecture, the threshold value $\bar{\sigma}$ of the coupling strength scales with $\lambda_2$, but the bound on the product of the coupling strength and delay scales with $\lambda_k$, the largest eigenvalue of the Laplacian. In this sense our result generalizes the Wu–Chua conjecture.

We will end with some remarks on a condition for local synchronization in networks with coupling of the form $u_i(t) = \sigma \sum_{j\in\mathcal{N}_i} a_{ij}(y_j(t-\tau) - y_i(t))$. As mentioned in the Introduction, this type of coupling will generally not vanish when the systems are synchronized. Therefore, to ensure existence of the synchronization manifold in a general sense, one wishes to assume that $\sum_{j\in\mathcal{N}_i} a_{ij} = 1$ for all $j$. Then the synchronized solution of the network is a solution of the system

$$\dot{\xi}(t) = f(\xi(t)) - \sigma BC[\xi(t) - \xi(t-\tau)].$$

(Note that for the type of time-delayed diffusive coupling that we considered the synchronized solution is a solution of the system $\dot{\xi}(t) = f(\xi(t))$.) There are several difficulties in establishing conditions for local synchronization as in Theorem 5. The first (and probably main) difficulty is that one has to determine the values of $\sigma$ and $\tau$ for which the system above has an asymptotically stable attractor. Note that this is an attractor in $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ and not in $\mathbb{R}^n$ as in our case! Next one has to investigate the uniform asymptotic stability of the linear variational systems

$$\dot{\eta}_j(t) = J_f(t)\eta_j(t) - \sigma BC\left(\eta(t) - \lambda_j(A)\eta(t-\tau)\right), \quad j = 2, \dots, k.$$

Here $\lambda_j(A)$ are the eigenvalues of the adjacency matrix that are smaller than 1. (When the graph is connected, the Laplacian matrix has a simple zero eigenvalue and the adjacency matrix has a simple eigenvalue $\lambda_1 = 1$. Also the moduli of all other eigenvalues are smaller than or equal to 1.) Note however that the stability of these systems depend on the values of $\tau$, $\sigma$ and $\sigma\lambda_j(A)$, while in the case we considered the stability of the linear variational system is determined by $\tau$ and $\sigma\lambda_j(L)$. The fact that in our case, for fixed $j$, the stability properties depend only on two parameters allowed us to introduce the local synchronization diagram $\mathcal{S}$ in the $(\sigma, \tau)$-plane and rephrase the conditions for stability of all $j$ as taking the intersection of the scaled copies $\mathcal{S}_j$. The fact that the stability problem for coupling of the form $u_i(t) = \sigma \sum_{j\in\mathcal{N}_i} a_{ij}(y_j(t-\tau) - y_i(t))$ depends, for fixed $j$, the three parameters $\tau$, $\sigma$ and $\sigma\lambda_j(A)$ means that once again the stability diagram can be obtained as an intersection of scaled copies of one diagram. However, this now needs to be done in three-dimensional space, which obviously makes it more difficult to visualize this.

**Acknowledgments**

The work of the second author was performed in the framework of the Belgian Program on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture, the Optimization in Engineering Centre OPTEC, the project STRT1-09/33 of the K.U.Leuven Research Foundation, and the project G.0712.11N of the Research Foundation—Flanders (FWO).

## Appendix A. Convex polytopic approximation and a quadratic Lyapunov functional for linear time-delay systems

Consider the linear system

$$\dot{\xi}(t) = G_1(t)\xi(t) + G_2\xi(t-\tau) \tag{A.1}$$

with constant $\tau > 0$, $\xi(t) \in \mathbb{R}^n$, matrix valued function $G_1 : \mathbb{R} \to \mathbb{R}^{n\times n}$ and constant matrix $G_2 \in \mathbb{R}^{n\times n}$. It is assumed that each entry of $G_1(t)$ is bounded for all $t$.

We are interested in the uniform asymptotic stability of the zero solution of (A.1). It is known that if the zero solution is uniformly asymptotically stable, then there exists a quadratic functional $V$, a Lyapunov functional, with a negative definite derivative along solutions of the system, cf. [48]. Conversely, if we can find a (quadratic) Lyapunov functional with negative definite derivative along solutions we can conclude uniform asymptotic stability of the zero solution. Sufficient conditions for uniform asymptotic stability via quadratic Lyapunov functionals for system (A.1) are presented in, for instance, [66,67]. We apply here a result of [67].

Consider the functional

$$V(\xi_t) = \xi^\top(t)P_0\xi(t) + \int_{t-\tau}^{t} \xi^\top(s)P_1\xi(s)\mathrm{d}s + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{\xi}^\top(s)P_2\dot{\xi}(s)\mathrm{d}s\mathrm{d}\theta$$

with positive definite matrices $P_0, P_1, P_2 \in \mathbb{R}^{n\times n}$. Then

$$\dot{V}(\xi_t) \le \frac{1}{\tau}\int_{t-\tau}^{t} \begin{pmatrix} \xi(t) \\ \xi(t-\tau) \\ \dot{\xi}(s) \end{pmatrix}^\top Q(t) \begin{pmatrix} \xi(t) \\ \xi(t-\tau) \\ \dot{\xi}(s) \end{pmatrix} \mathrm{d}s$$

with

$$Q(t) = \begin{pmatrix} (P_0G_1 + G_1^\top P_0 + \tau G_1^\top P_2 G_1)(t) + Y + Y^\top + P_1 & \star & \star \\ G_2^\top P_0 Y^\top + W & -W - W^\top + \tau G_2^\top P_2 G_2 - P_1 & \star \\ -\tau Y^\top & -\tau W^\top & -\tau P_2 \end{pmatrix},$$

with matrices $Y, W \in \mathbb{R}^{n\times n}$ and $\star$ denoting the symmetric part. Using a Schur complement, cf. [36], it may be shown that the symmetric matrix $Q(t) < 0$ for all $t$ if and only if

$$\bar{Q}(t) = \begin{pmatrix} (P_0G_1 + G_1^\top P_0)(t) + Y + Y^\top + P_1 & \star & \star & \star \\ G_2^\top P_0 Y^\top + W + \tau G_1^\top(t) P_2 G_2 & -W - W^\top - P_1 & \star & \star \\ -\tau Y^\top & -\tau W^\top & -\tau P_2 & \star \\ -\tau P_2 G_1^\top(t) & -\tau P_2 G_2 & \mathbf{0} & -\tau P_2 \end{pmatrix} < 0$$

for all $t$. Finding the matrices $P_1, P_2, P_3, Y, W$ such that the matrix $\bar{Q}(t) < 0$ for all $t$ can be a difficult task. Note however that $G_1(t)$ bounded for all $t$. This means that there exist an integer $\bar{j} \ge 1$ and matrices $\bar{G}_j$ ($j = 1, \ldots, \bar{j}$) such that $G_1(t) \in \mathrm{Co}\{\bar{G}_1, \ldots, \bar{G}_\ell\}$ for all $t$. Here Co denotes the convex hull. Hence we can write, for each $t$,

$$G_1(t) = \sum_{j=1}^{\bar{j}} \nu_j(t)\bar{G}_j, \qquad \sum_{j=1}^{\bar{j}} \nu_j(t) = 1.$$

Moreover, since $\bar{Q}(t)$ is affine in $G_1(t)$, we have $\bar{Q}(t) < 0$ for all $t$ if the *constant* matrices

$$\begin{pmatrix} P_0\bar{G}_j + \bar{G}_j^\top P_0 + Y + Y^\top + P_1 & \star & \star & \star \\ G_2^\top P_0 Y^\top + W + \tau\bar{G}_j^\top P_2 G_2 & -W - W^\top - P_1 & \star & \star \\ -\tau Y^\top & -\tau W^\top & -\tau P_2 & \star \\ -\tau P_2 \bar{G}_j^\top & -\tau P_2 G_2 & \mathbf{0} & -\tau P_2 \end{pmatrix} < 0, \tag{A.2}$$

for all $j = 1, \ldots, \ell$. Solutions to the latter problem, i.e. finding positive definite matrices $P_0, P_1, P_2$ and matrices $Y, W$ such that the inequalities (A.2) are satisfied, can be found with various software packages such as *SeDuMi* [68] or the *Matlab LMI solver* (part of the Robust Control Toolbox).

## Appendix B. Proofs and auxiliary results

### *B.1. Results on boundedness*

**Theorem 11** (*Theorem 4.2.10 of [39]*)**.** *Consider the Retarded Functional Differential Equation (RFDE)*

$$\dot{x}(t) = f(t, x_t),$$

*where $f : \Omega \subset \mathbb{R} \times \mathcal{C}([-h, 0], \mathbb{R}^k) \to \mathbb{R}^k$ is continuous and takes bounded sets into bounded sets. Suppose there exist $w_1, w_2, w_3, w_4 \in \mathcal{K}_\infty$ and a continuous functional $V : \mathbb{R} \times \mathcal{C}([-h, 0], \mathbb{R}^k) \to \mathbb{R}$ that is locally Lipschitz in $\phi$ for each $t \in [t_0, \infty)$ and $\|\phi\| < \infty$, such that*

$$w_1(|\phi(0)|) \le V(t,\phi) \le w_2(|\phi(0)|) + w_3\left(\int_{-h}^{0} w_4(|\phi(s)|)\mathrm{d}s\right)$$

*and*

$$\dot{V}(t,\phi) \le -w_4(|\phi(0)|) + M$$

*for some $M > 0$. Then the solutions of the RFDE are uniformly bounded and uniformly ultimately bounded.*

**Remark 4.** It follows from the proof of the theorem above that for any $B_1 > 0$ such that $\|\phi\| \le B_1$ we have $|x(t; t_0, \phi)| < B_2$ with

$$B_2 = \max\{w_1^{-1}([w_2(B_1) + w_3(hw_4(B_1)) + hM]), \\ w_1^{-1}([w_2(w_4^{-1}(M)) + w_3(hM)])\}.$$

The following theorem is a slightly modified version of Theorem 4 in [42].

**Theorem 12.** *Consider the autonomous RFDE*

$$\dot{x}(t) = f(x_t),$$

*where $f : \Omega \subset \mathcal{C}([-h, 0], \mathbb{R}^k) \to \mathbb{R}^k$ is continuous and takes bounded sets into bounded sets. Suppose that there is a continuous non-decreasing function $K : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\phi \in \mathcal{C}, \quad \|\phi\| \le H \Rightarrow |f(\phi)| \le K(H),$$

*and suppose in addition that there exist strictly increasing functions $w_1, w_2, w_3 \in \mathcal{K}_\infty$ and a functional $V : \mathcal{C}([-h, 0], \mathbb{R}^k) \to \mathbb{R}_+$ such that*

$$w_1(|\phi(0)|) \le V(\phi) \le w_2(|\phi(0)|) + w_3(\|\phi\|).$$

*If there exist three positive constants $B_1, B^*, B_2$ such that*

i. $w_2(B_1) + w_3(B^*) < w_1(B_2)$,
ii. $B^* - B_1 > hK(B_2)$,
iii. $\dot{V}(\phi) \leq 0$ *for* $B_1 \leq |\phi(0)| \leq \|\phi\| \leq B_2$,

*then* $\|\phi\| \leq B_1$ *implies* $|x(t; t_0, \phi)| < B_2$ *for all* $t \geq t_0$.

**Proof.** There is no problem proving the theorem with

iv. $\dot{V}(\phi) \leq 0 \quad$ for $B_1 \leq |\phi(0)|$

instead of iii since we show that the conditions imply $|x(t)| < B_2$ for all $t \geq t_0$, and hence can restrict $\dot{V}(\phi) \leq 0$ on $B_1 \leq |\phi(0)| \leq \|\phi\| \leq B_2$. So the proof is essentially a repetition of the proof given by Burton [42].

We prove that whenever $|x(t)| = B_1$, then $V(x_t) \leq w_2(B_1) + w_3(B^*)$. Assume $\|\phi\| < B_1$ and let $t_1$ and $t_2$ be such that $t_0 \leq t_1 < t_2$, $|x(t_1)| = B_1$ and $|x(t)| > B_1$ on $(t_1, t_2)$. Then for $t_1 \leq t < t_2$

$$w_1(|x(t)|) \leq V(x_t) \leq V(x_{t_1}) \leq w_2(B_1) + w_3(\|x_{t_1}\|) \leq w_2(B_1) + w_3(B^*)$$

by i and iv. Then i implies $|x(t)| < B_2$ on $[t_0, t_2)$. Since $V$ cannot increase as long as $|x(t)| > B_1$ the case of interest is when $|x(t_2)| = B_1$ and there are $t_3, t_4, t_4 > t_3 \geq t_2$ for which $|x(t)| > B_1$ on $(t_3, t_4)$ and $|x(t)| \leq B_1$ on $[t_2, t_3]$. Since we may have $\dot{V}(x_t) > 0$ on $(t_2, t_3)$, we need to show that $V(x_{t_3}) \leq w_2(B_1) + w_3(B^*)$, i.e. $\|x_{t_3}\| \leq B^*$. Condition ii implies that it takes more than $h$ time-units to move from $B^*$ to $B_1$ (and vice versa), hence $\|x_{t_2}\| < B^*$ and thus $\|x_{t_3}\| < B^*$, which implies

$$w_1(|x(t_3)|) \leq V(x_{t_3}) \leq w_2(B_1) + w_3(B^*) < w_1(B_2),$$

i.e. $|x(t_3)| < B_2$. Clearly, for $t_3 \leq t < t_4$ condition iv implies

$$w_1(|x(t)|) \leq V(x_t) \leq V(x_{t_3}) \leq w_2(B_1) + w_3(B^*) < w_1(B_2).$$

A repetition of the arguments used above shows that we always have $|x(t)| < B_2$. $\square$

### B.2. Proof of Theorem 3

Let $x(t) = \operatorname{col}(x_1(t), \ldots, x_k(t))$ and $F(x(t)) = \operatorname{col}\big(f(x_1(t)), \ldots, f(x_k(t))\big)$ such that we can write (1), (2) as

$$\dot{x}(t) = F(x(t)) - \sigma(L \otimes BC)x(t - \tau).$$

We have to consider two cases: $\sigma < 1$ and $\sigma \geq 1$. We start with the latter case. We rescale time $t^* = \sigma t$ such that

$$x'(t^*) = \frac{1}{\sigma}F(x(t^*)) - (L \otimes BC)x(t^* - \sigma\tau) =: \tilde{F}(x_{t^*})$$

where $'$ denotes the (right-hand) derivative with respect to $t^*$. Note that the state-space of the coupled system in rescaled time is $\mathcal{C}([-\sigma\tau, 0], \mathbb{R}^{kn})$. Consider the functional

$$V(x_{t^*}) = S(x_1(t^*)) + \cdots + S(x_k(t^*)) + \int_{t^*-\sigma\tau}^{t^*} x^\top(s)P_1x(s)\mathrm{d}s + \int_{-\sigma\tau}^{0}\mathrm{d}\theta \int_{t^*+\theta}^{t^*} x'^\top(s)P_2x'(s)\mathrm{d}s$$

with positive semi-definite matrices $P_1, P_2$ to be determined later and $S$ is the storage function. Then

$$\begin{aligned} V'(x_{t^*}) \leq{} & -\tfrac{1}{\sigma}H(x_1(t^*)) - \cdots - \tfrac{1}{\sigma}H(x_k(t^*)) \\ & - x^\top(t^*)(L \otimes C^\top C)x(t^* - \sigma\tau) + x^\top(t^*)P_1x(t^*) \\ & - x^\top(t^* - \sigma\tau)P_1x(t^* - \sigma\tau) + \sigma\tau x'^\top(t^*)P_2x'(t^*) \\ & - \int_{t^*-\sigma\tau}^{t^*} x'^\top(s)P_2x'(s)\mathrm{d}s. \end{aligned}$$

Using Jensen's inequality [69] we derive

$$-\int_{t^*-\sigma\tau}^{t^*} x'^\top(s)P_2x'(s)\mathrm{d}s \leq -\frac{1}{\sigma\tau}(x(t^*) - x(t^* - \sigma\tau))^\top P_2(x(t^*) - x(t^* - \sigma\tau))$$

which gives that

$$\begin{aligned} \dot{V}(x_t) \leq{} & -\tfrac{1}{\sigma}H(x_1(t^*)) - \cdots - \tfrac{1}{\sigma}H(x_k(t^*)) \\ & + \tfrac{\tau}{\sigma}F^\top(x(t^*))P_2F(x(t^*)) \\ & + \tau\left(x^\top(t^* - \rho)(L \otimes BC)^\top P_2F(x(t^*))\right. \\ & \left. + F^\top(x(t^*))P_2(L \otimes BC)x(t^* - \rho)\right) - \begin{pmatrix} x(t^*) \\ x(t^* - \rho) \end{pmatrix}^\top \\ & \times \begin{pmatrix} \frac{1}{\rho}P_2 - P_1 & \frac{1}{2}(L \otimes C^\top C) - \frac{1}{\rho}P_2 \\ \frac{1}{2}(L \otimes C^\top C) - \frac{1}{\rho}P_2 & P_1 + \frac{1}{\rho}P_2 - \rho(L \otimes BC)^\top P_2(L \otimes BC) \end{pmatrix} \\ & \times \begin{pmatrix} x(t^*) \\ x(t^* - \rho) \end{pmatrix} \end{aligned}$$

where $\rho = \sigma\tau$. Choose $P_1 = P_2 = \frac{1}{2}L \otimes C^\top C$ and observe that, using $CB = I_m$,

$$\begin{aligned} &\begin{pmatrix} \frac{1}{\rho}P_2 - P_1 & \frac{1}{2}(L \otimes C^\top C) - \frac{1}{\rho}P_2 \\ \frac{1}{2}(L \otimes C^\top C) - \frac{1}{\rho}P_2 & P_1 + \frac{1}{\rho}P_2 - \rho(L \otimes BC)^\top P_2(L \otimes BC) \end{pmatrix} \\ &= \begin{pmatrix} (\frac{1}{\rho} - 1)I_{km} & (1 - \frac{1}{\rho})I_{km} \\ (1 - \frac{1}{\rho})I_{km} & (1 + \frac{1}{\rho})I_{km} - \rho(L^2 \otimes I_m) \end{pmatrix} \otimes \tfrac{1}{2}(L \otimes C^\top C). \end{aligned}$$

By the assumption that the row sums of $A$ are not larger than 1, we have that $\|L^2\| \leq 4$. Hence we find

$$\begin{aligned} &\begin{pmatrix} \frac{1}{\rho}P_2 - P_1 & \frac{1}{2}(L \otimes C^\top C) - \frac{1}{\rho}P_2 \\ \frac{1}{2}(L \otimes C^\top C) - \frac{1}{\rho}P_2 & P_1 + \frac{1}{\rho}P_2 - \rho(L \otimes BC)^\top P_2(L \otimes BC) \end{pmatrix} \\ &\leq \begin{pmatrix} \frac{1}{\rho} - 1 & 1 - \frac{1}{\rho} \\ 1 - \frac{1}{\rho} & 1 + \frac{1}{\rho} - 4\rho \end{pmatrix} \otimes \tfrac{1}{2}(L \otimes C^\top C) =: Q. \end{aligned}$$

Since $L \geq 0$ we find that $Q \geq 0$ provided that $\rho \leq \frac{1}{2}$. Thus for $\rho \leq \frac{1}{2}$

$$\begin{aligned} V'(x_{t^*}) \leq{} & -\tfrac{1}{\sigma}H(x_1(t^*)) - \cdots - \tfrac{1}{\sigma}H(x_k(t^*)) \\ & + \tfrac{\tau}{2\sigma}F^\top(x(t^*))(L \otimes C^\top C)F(x(t^*)) \\ & + \tfrac{1}{2}\tau\left(x^\top(t^* - \rho)(L^2 \otimes C^\top C)F(x(t^*))\right. \\ & \left. + F^\top(x(t^*))(L^2 \otimes C^\top C)x(t^* - \rho)\right). \end{aligned}$$

As in the proof of Theorem 1, the strict semi-passivity property implies that there are constants $\kappa > 0$ and $R_1 > R$, such that if $|x(t)| \geq R_1$, then

$$\begin{aligned} \sigma V'(x_{t^*}) \leq{} & -\kappa x^\top(t^*)(I_k \otimes C^\top C)x(t^*) \\ & + \tfrac{1}{2}\tau F^\top(x(t^*))(L \otimes C^\top C)F(x(t^*)) \\ & + \tfrac{1}{2}\rho\left(x^\top(t^* - \rho)(L^2 \otimes C^\top C)F(x(t^*))\right. \\ & \left. + F^\top(x(t^*))(L^2 \otimes C^\top C)x(t^* - \rho)\right). \end{aligned}$$

Clearly, for bounded $|F(x(t^*))|$ and $|x(t^* - \rho)|$ and both $\tau$ and $\rho$ sufficiently small we have $V'(x_{t^*}) \leq 0$. We determine the bounds and $\tau$ and $\rho$ with the help of Theorem 12. Pick $B_1 \geq R_1$ and $B^* > B_1$. We need to determine the functions $w_1, w_2, w_3$ and the constant $B_2$. Since the storage function satisfies $s_1(|x_i(t^*)|) \leq S(x_i(t^*)) \leq s_2(|x(t^*)|)$, $s_1, s_2 \in \mathcal{K}_\infty$, we can clearly find $\bar{v}_1, \bar{v}_2 \in \mathcal{K}_\infty$ such that $\bar{v}_1(|x(t^*)|) \leq S(x_1(t^*)) + \cdots + S(x_k(t^*)) \leq \bar{v}_2(|x(t^*)|)$. (Note that if $S$ is quadratic then we can choose, $\bar{v}_1 = s_1$, $\bar{v}_2 = s_2$.) Then

$$\bar{v}_1(|x(t^*)|) = w_1(|x(t^*)|) \leq V(x_{t^*})$$

and

$$V(x_{t^*}) \leq \bar{v}_2(|x(t^*)|) + \rho \|x_{t^*}\|^2 + \rho \left|\tilde{F}(x_{t^*})\right|^2.$$

Since $B_2 > B^* > B_1$ and we have to satisfy condition ii in Theorem 12 we have

$$V(x_{t^*}) \leq \bar{v}_2(|x(t^*)|) + \rho \|x_{t^*}\|^2 + (B^* - B_1) \left|\tilde{F}(x_{t^*})\right|$$

as long as $\|x_{t^*}\| \leq B_2$. Since the functions $f$ in (1) are assumed to be sufficiently smooth, we can find a non-decreasing are assumed to be sufficiently smooth, function $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|F(x(t^*))| \leq \ell(|x(t^*)|)\, |x(t^*)|$$

hence

$$\left|\tilde{F}(x_{t^*})\right| \leq \tfrac{1}{\sigma} |F(x(t^*))| + 2\|x_{t^*}\| \leq \tfrac{1}{\sigma}\ell(|x(t^*)|)\, |x(t^*)| + 2\|x_{t^*}\|.$$

Then, for $\rho < \frac{1}{2}$ and because $\sigma \geq 1$ we find

$$V(x_{t^*}) \leq w_2(|x(t^*)|) + w_3(\|x_{t^*}\|)$$

with

$$w_2(|x(t^*)|) = \bar{v}_2(|x(t^*)|) + (B^* - B_1)\ell(|x(t^*)|)\, |x(t^*)|$$
$$w_3(\|x_{t^*}\|) = \tfrac{1}{2}\|x_{t^*}\|^2 + 2(B^* - B_1)\|x_{t^*}\|.$$

We can now find the number $B_2$ and subsequently pick $\rho$, $\rho \leq \frac{1}{2}$ and $\tau$ small enough to ensure that condition ii of Theorem 12 is satisfied. Then we pick $\tau_{\max}$ and $\rho$ small enough such that

$$\kappa B_1^2 \geq \tau_{\max} F_{\max}^2(B_2) + 4\rho B_2^2, \tag{B.3}$$

with constant

$$F_{\max}(B_2) \geq |F(x(t))|, \quad \forall |x(t)| \leq B_2.$$

If the inequality (B.3) holds we have $V'(x_{t^*}) \leq 0$.

The case for $\sigma < 1$ is completely analogous to the case considered above. We consider the functional

$$\begin{aligned} V(x_t) &= S(x_1(t)) + \cdots + S(x_k(t)) \\ &\quad + \frac{1}{2}\sigma \int_{t-\tau}^{t} x^\top(s)(L \otimes C^\top C)x(s)\mathrm{d}s \\ &\quad + \frac{1}{2}\int_{-\tau}^{0}\mathrm{d}\theta \int_{t+\theta}^{t} \dot{x}^\top(s)(L \otimes C^\top C)\dot{x}(s)\mathrm{d}s \end{aligned}$$

and we can show that

$$\begin{aligned} \dot{V}(x_t) &\leq -\kappa x^\top(t)(I_k \otimes C^\top C)x(t) + \tfrac{1}{2}\tau F^\top(x(t))(L \otimes C^\top C)F(x(t)) \\ &\quad + \tfrac{1}{2}\rho\left(x^\top(t-\tau)(L^2 \otimes C^\top C)F(x(t))\right. \\ &\quad \left. + F^\top(x(t))(L^2 \otimes C^\top C)x(t-\tau)\right) \end{aligned}$$

for $\rho \leq \frac{1}{2}$. Note that the upper bound for $\dot{V}(x_t)$ is identical to the derived upper bound for $\sigma V'(x_{t^*})$. In addition,

$$w_1(|x(t)|) \leq V(x_t) \leq w_2(|x(t)|) + w_3(\|x_t\|)$$

with the functions $w_1, w_2, w_3$ given above. Then we may choose the constant $B_2$ the same and it follows readily that conditions ii and iii of Theorem 12 are satisfied for the $\rho$ and $\tau_{\max}$ as determined for the case $\sigma \geq 1$.

### B.3. Proof of Theorems 9 and 10

We need the following result.

**Lemma 13.** *Let*

$$V_0((z_i - z_j)(t)) = \frac{1}{2}(z_i - z_j)^\top(t)P(z_i - z_j)(t),$$

*with $P$ as in Lemma 8. If the conditions of Lemma 8 are satisfied, then there is a constant $c_1 > 0$ such that*

$$\begin{aligned} \dot{V}_0\big|_{y_i(t)=y_j(t)=y(t)} &= (z_i - z_j)^\top(t)P[q(z_i(t), y(t)) - q(z_j(t), y(t))] \\ &\leq -c_1 \left|(z_i - z_j)(t)\right|^2. \end{aligned}$$

**Proof.** Fix $z_i, z_j$ and $y$ and denote

$$\Phi(\zeta) = (z_i - z_j)^\top P[q(\zeta(z_i - z_j) + z_j, y)]$$

with $\zeta \in [0, 1]$. Then

$$(z_i - z_j)^\top P[q(z_i, y) - q(z_j, y(t))] = \Phi(1) - \Phi(0).$$

Since $q$ is assumed to be sufficiently smooth we can apply the mean value theorem such that

$$\Phi(1) - \Phi(0) = \frac{\partial \Phi}{\partial \zeta}(\zeta^*) = (z_i - z_j)^\top P\left[\frac{\partial q}{\partial z_i}(z^*, y)\right](z_i - z_j),$$

with some $\zeta^* \in [0, 1]$ and $z^* = \zeta^*(z_i - z_j) + z_j$. By assumption the symmetrized matrix

$$\tfrac{1}{2}\left(\left[\frac{\partial q}{\partial z_i}(z^*, y)\right]^\top P + P\left[\frac{\partial q}{\partial z_i}(z^*, y)\right]\right)$$

is uniformly negative definite, that is, there is a positive constant $c_1$ such that for all $z^*$ and $y$ the matrix

$$c_1 I_{n-m} + \tfrac{1}{2}\left(\left[\frac{\partial q}{\partial z_i}(z^*, y)\right]^\top P + P\left[\frac{\partial q}{\partial z_i}(z^*, y)\right]\right)$$

is negative semi-definite. □

We now prove Theorem 10. Let

$$\tilde{z}(t) = \begin{pmatrix} z_1(t) - z_2(t) \\ \vdots \\ z_1(t) - z_k(t) \end{pmatrix} \quad \text{and} \quad \tilde{y}(t) = \begin{pmatrix} y_1(t) - y_2(t) \\ \vdots \\ y_1(t) - y_k(t) \end{pmatrix}.$$

We show that the conditions in the theorem imply $\lim_{t\to\infty} \tilde{z}(t) = 0$ and $\lim_{t\to\infty} \tilde{y}(t) = 0$. Consider the function

$$V(\tilde{z}(t), \tilde{y}(t)) = \tilde{z}^\top(t)P_1\tilde{z}(t) + \tilde{y}^\top(t)P_2\tilde{y}(t),$$

with positive definite matrices $P_1, P_2$ of appropriate dimension. We prove our result with the help of the Lyapunov–Razumikhin theorem, cf. [38] Section 5.4 Theorem 4.2. In particular we show that for $(\sigma, \tau) \in \mathcal{S}_2^* \cap \mathcal{S}_k^*$ there exist constants $\epsilon > 0$ and $\gamma > 1$ such that $\dot{V}(\tilde{z}(t), \tilde{y}(t)) \leq -\epsilon(|\tilde{z}(t)|^2 + |\tilde{y}(t)|^2)$ whenever $\gamma^2 V(\tilde{z}(t), \tilde{y}(t)) > V(\tilde{z}(t+\theta), \tilde{y}(t+\theta)), \theta \in [-2\tau, 0]$.

Consider

$$\tilde{z}^\top(t)P_1\tilde{z}(t) =: V_1(\tilde{z}(t)).$$

Take $P_1 = \frac{1}{2}I_{k-1} \otimes P$ with the matrix $P$ as in Lemma 8. Without loss of generality we assume $\|P\| = 1$. Then

$$V_1(\tilde{z}(t)) = \sum_{j=2}^{k}(z_1(t) - z_j(t))^\top P(z_1(t) - z_j(t))$$

and

$$\dot{V}_1(\tilde{z}(t),\tilde{y}(t)) = \sum_{j=2}^{k}(z_1(t)-z_j(t))^\top P(q(z_1(t),y_1(t)) - q(z_j(t),y_j(t))) = \sum_{j=2}^{k}(z_1(t)-z_j(t))^\top P(q(z_1(t),y_1(t)) - q(z_j(t),y_1(t))) + (z_1(t)-z_j(t))^\top P(q(z_j(t),y_1(t)) - q(z_j(t),y_j(t))).$$

Invoking Lemma 13 we conclude that

$$\dot{V}_1(\tilde{z}(t)) \le \sum_{j=2}^{k} -c_1(z_1(t)-z_j(t))^\top(z_1(t)-z_j(t)) + (z_1(t)-z_j(t))^\top P(q(z_j(t),y_1(t)) - q(z_j(t),y_j(t))).$$

Because the function $q$ is assumed to be sufficiently smooth, it is (locally) Lipschitz in both arguments. Thus there is a positive constant $c_2$ such that

$$q(z_j(t),y_1(t)) - q(z_j(t),y_j(t)) \le c_2\left|y_i(t)-y_j(t)\right|,$$

hence

$$(z_1(t)-z_j(t))^\top P(q(z_j(t),y_1(t)) - q(z_j(t),y_j(t))) \le c_2|z_1(t)-z_j(t)|\,|y_1(t)-y_j(t)|,$$

and

$$\dot{V}_1(\tilde{z}(t),\tilde{y}(t)) \le \sum_{j=2}^{k} -c_1|z_1(t)-z_j(t)|^2 + c_2|z_1(t)-z_j(t)|\,|y_1(t)-y_j(t)|.$$

Note that the constants $c_1$ and $c_2$ only depend on the function $q$ and the bounds on the variables $z_i(t)$ and $y_i(t)$.

Now consider

$$\dot{\tilde{y}}(t) = \begin{pmatrix} a(z_1(t),y_1(t)) - a(z_2(t),y_2(t)) \\ \vdots \\ a(z_1(t),y_1(t)) - a(z_k(t),y_k(t)) \end{pmatrix} - \sigma(\tilde{L}\otimes I_m)\tilde{y}(t-\tau)$$

where $\tilde{L} = \begin{pmatrix}\mathbf{1} & -I_{k-1}\end{pmatrix} L \begin{pmatrix}\mathbf{1} & -I_{k-1}\end{pmatrix}^+$, and $M^+$ denotes the Moore–Penrose pseudo-inverse of matrix $M$. Note that the eigenvalues of $\tilde{L}$ are the nonzero eigenvalues of $L$. Since $L$ is symmetric there exists a non-singular matrix $U$, $\|U\| = 1$, such that

$$U\tilde{L}U^{-1} = \begin{pmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} =: \Lambda.$$

Let $P_1 = \frac{1}{2}U^\top U \otimes I_m$. Then

$$\dot{V}_2(\tilde{z}(t),\tilde{y}(t)) = \tilde{y}^\top(t)(U^\top U\otimes I_m) \times \begin{pmatrix} a(z_1(t),y_1(t)) - a(z_2(t),y_2(t)) \\ \vdots \\ a(z_1(t),y_1(t)) - a(z_k(t),y_k(t)) \end{pmatrix} - \sigma\tilde{y}^\top(t)(U^\top\otimes I_m)\underbrace{(U\otimes I_m)(\tilde{L}\otimes I_m)(U^{-1}\otimes I_m)}_{=\Lambda\otimes I_m} \times (U\otimes I_m)\tilde{y}(t-\tau).$$

Consider $a(z_1(t),y_1(t)) - a(z_j(t),y_j(t))$, $j = 2,\dots,k$, and add and subtract $a(z_1(t),y_j(t))$ to obtain

$$a(z_1(t),y_1(t)) - a(z_1(t),y_j(t)) + a(z_1(t),y_j(t)) - a(z_j(t),y_j(t)).$$

Following [63], we let

$$a(z_1(t),y_1(t)) - a(z_1(t),y_j(t)) = \left[\int_0^1 \frac{\partial a}{\partial y}(z_1(t),\xi y_1(t)+(1-\xi)y_j(t))d\xi\right](y_1(t)-y_j(t)) =: D_ya(z_1(t),y_1(t),y_j(t))(y_1(t)-y_j(t))$$

which gives that

$$\begin{pmatrix} a(z_1(t),y_1(t)) - a(z_2(t),y_2(t)) \\ \vdots \\ a(z_1(t),y_1(t)) - a(z_k(t),y_k(t)) \end{pmatrix} = \begin{pmatrix} a(z_1(t),y_j(t)) - a(z_2(t),y_j(t)) \\ \vdots \\ a(z_1(t),y_j(t)) - a(z_k(t),y_j(t)) \end{pmatrix} + \begin{pmatrix} D_ya(z_1(t),y_1(t),y_2(t)) & & \\ & \ddots & \\ & & D_ya(z_1(t),y_1(t),y_k(t)) \end{pmatrix}\tilde{y}(t).$$

Since all $z_i(t)$ and $y_i(t)$ are bounded, the matrix $D_ya(z_1(t),y_1(t),y_j(t))$ is bounded, which means that there exists a constant $c_3 \ge 0$ such that

$$(y_1(t)-y_j(t))^\top D_ya(z_1(t),y_1(t),y_j(t))(y_1(t)-y_j(t)) \le c_3\left|y_1(t)-y_j(t)\right|^2.$$

Then

$$\tilde{y}^\top(t)(U^\top U\otimes I_m)\begin{pmatrix} D_ya(z_1(t),y_1(t),y_2(t)) & & \\ & \ddots & \\ & & D_ya(z_1(t),y_1(t),y_k(t)) \end{pmatrix} \times\tilde{y}(t) \le c_3\left|\tilde{y}(t)\right|^2,$$

since the eigenvalues of the symmetric matrix

$$\frac{1}{2}(U^\top U\otimes I_m)\begin{pmatrix} D_ya(z_1(t),y_1(t),y_2(t)) & & \\ & \ddots & \\ & & D_ya(z_1(t),y_1(t),y_k(t)) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} D_ya(z_1(t),y_1(t),y_2(t)) & & \\ & \ddots & \\ & & D_ya(z_1(t),y_1(t),y_k(t)) \end{pmatrix}(U^\top U\otimes I_m)$$

can be uniformly bounded by the constant $c_3$. Since $\|\cdot\|$ is an induced norm, we have that $\|U\| = 1$ implies that $\left\|U^\top U\right\| = 1$. Therefore,

$$\tilde{y}^\top(t)(U^\top U\otimes I_m)\begin{pmatrix} a(z_1(t),y_j(t)) - a(z_2(t),y_j(t)) \\ \vdots \\ a(z_1(t),y_j(t)) - a(z_k(t),y_j(t)) \end{pmatrix} \le c_4\left|\tilde{z}(t)\right|\,\left|\tilde{y}(t)\right|,$$

where $c_4$ is a local Lipschitz constant of the function $a$ with respect to its first argument. We conclude

$$\dot{V}_2(\tilde{y}(t)) \le c_3\left|\tilde{y}(t)\right|^2 + c_4\left|\tilde{z}(t)\right|\,\left|\tilde{y}(t)\right| - \sigma((U\otimes I_m)\tilde{y}(t))^\top(\Lambda\otimes I_m)((U\otimes I_m)\tilde{y}(t-\tau)).$$

For $t \ge t_0 + \tau$ we may write

$$\tilde{y}(t-\tau) = \tilde{y}(t) - \int_{t-\tau}^{t}\dot{\tilde{y}}(s)\mathrm{d}s,$$

hence we have

$$-\sigma((U\otimes I_m)\tilde{y}(t))^\top(\Lambda\otimes I_m)((U\otimes I_m)\tilde{y}(t-\tau))$$
$$=-\sigma((U\otimes I_m)\tilde{y}(t))^\top(\Lambda\otimes I_m)((U\otimes I_m)\tilde{y}(t))$$
$$+\sigma\int_{-\tau}^{0}((U\otimes I_m)\tilde{y}(t))^\top(\Lambda\otimes I_m)(U\otimes I_m)\dot{\tilde{y}}(t+s)\mathrm{d}s.$$

Clearly

$$-\sigma((U\otimes I_m)\tilde{y}(t))^\top(\Lambda\otimes I_m)((U\otimes I_m)\tilde{y}(t))\le -\sigma\lambda_2\left|\tilde{y}(t)\right|,$$

and

$$\int_{-\tau}^{0}(U\otimes I_m)\dot{\tilde{y}}(t+s)\mathrm{d}s=\int_{-\tau}^{0}(U\otimes I_m)$$
$$\times\begin{pmatrix}a(z_1(t+s),y_1(t+s))-a(z_2(t+s),y_2(t+s))\\ \vdots\\ a(z_1(t+s),y_1(t+s))-a(z_k(t+s),y_k(t+s))\end{pmatrix}$$
$$-\sigma(\tilde{\Lambda}\otimes I_m)(U\otimes I_m)\tilde{y}(t-\tau+s)\mathrm{d}s$$
$$\le\int_{-\tau}^{0}c_5\left|\tilde{y}(t+s)\right|+c_4\left|\tilde{z}(t+s)\right|$$
$$-\sigma(\tilde{\Lambda}\otimes I_m)(U\otimes I_m)\tilde{y}(t-\tau+s)\mathrm{d}s$$

with $c_5$ a Lipschitz constant of the function $a$ with respect to its second argument. If $\gamma\left|\tilde{y}(t)\right|>\left|\tilde{y}(t+\theta)\right|,\gamma\left|\tilde{z}(t)\right|>\left|\tilde{z}(t+\theta)\right|$, then

$$\sigma\int_{-\tau}^{0}((U\otimes I_m)\tilde{y}(t))^\top(\Lambda\otimes I_m)\times(c_5\left|\tilde{y}(t+s)\right|$$
$$+c_4\left|\tilde{z}(t+s)\right|-\sigma(\tilde{\Lambda}\otimes I_m)(U\otimes I_m)\tilde{y}(t-\tau+s)\mathrm{d}s)$$
$$\le\sigma\tau\lambda_k\gamma\left((c_5+\sigma\lambda_k)\left|\tilde{y}(t)\right|^2+c_4\left|\tilde{z}(t)\right|\,\left|\tilde{y}(t)\right|\right).$$

Thus, for $\gamma\left|\tilde{y}(t)\right|>\left|\tilde{y}(t+\theta)\right|,\gamma\left|\tilde{z}(t)\right|>\left|\tilde{z}(t+\theta)\right|$, $\theta\in[-2\tau,0]$, $\gamma>1$, we have

$$\dot{V}_2(\tilde{z}(t),\tilde{y}(t))\le\sigma\tau\lambda_k\gamma\,(c_3+(c_5+\sigma\lambda_k))\left|\tilde{y}(t)\right|^2$$
$$+c_4\,(1+\sigma\tau\lambda_k\gamma)\left|\tilde{z}(t)\right|\,\left|\tilde{y}(t)\right|.$$

Note that the constants $c_3, c_4, c_5$ depend only on the function $a$ and the bounds on $z_i(t), y_i(t)$.

Denote $\rho=\sigma\tau$ and consider the matrix

$$W=\begin{pmatrix}c_1 & -\dfrac{c_2+c_4(1+\rho\lambda_k\gamma)}{2}\\ -\dfrac{c_2+c_4(1+\rho\lambda_k\gamma)}{2} & \lambda_2\sigma-c_3-\rho\lambda_k\gamma(c_5+\sigma\lambda_k)\end{pmatrix}.$$

For sufficiently large $\sigma\lambda_2$ and sufficiently small $\rho\lambda_k$ there is a $\gamma>1$ for which $W$ is positive definite. Note that $W$ positive definite implies that $\dot{V}(\tilde{z}(t),y(t))$ is negative definite if $\gamma^2V(\tilde{z}(t),\tilde{y}(t))>V(\tilde{z}(t+\theta),\tilde{y}(t+\theta)),\theta\in[-2\tau,0]$,[1] hence $(\tilde{z}(t),\tilde{y}(t))\equiv(0,0)$ is a uniformly asymptotically stable equilibrium.

For $k=2$ coupled systems we have $\lambda_2=\lambda_k=2$. Then there are constants $\bar{\sigma}>0$ and $\bar{\rho}<\frac{1}{2}$ and a $\gamma>1$ such that the matrix

$$W=\begin{pmatrix}c_1 & -\dfrac{c_2+c_4(1+2\rho\gamma)}{2}\\ -\dfrac{c_2+c_4(1+2\rho\gamma)}{2} & 2\sigma-c_3-2\rho\gamma(c_5+2\sigma)\end{pmatrix}$$

[1] Note that for $\gamma\left|\tilde{z}(t)\right|>\left|\tilde{z}(t+\theta)\right|$ and or $\gamma\left|\tilde{y}(t)\right|>\left|\tilde{y}(t+\theta)\right|$ we have, since $V$ is quadratic, $\gamma^2V(\tilde{z}(t),\tilde{y}(t))>V(\tilde{z}(t+\theta),\tilde{y}(t+\theta))$.

is positive definite for $\sigma\ge\bar{\sigma}$ and $\sigma\tau\le\bar{\rho}$. This implies that two coupled systems globally synchronize for $(\sigma,\tau)\in\mathcal{S}^*\cap\mathcal{S}_M$, with $\mathcal{S}_M$ the set of values of $\sigma$ and $\tau$ for which the solutions of the coupled system are bounded, and

$$\mathcal{S}^*:=\{(\sigma,\tau)|\sigma>\bar{\sigma}>0\text{ and }\sigma\tau<\rho<\tfrac{1}{2}\}.$$

This proves Theorem 9. For $k>2$ systems we clearly have $W$ positive definite if $\sigma\lambda_2\ge2\bar{\sigma},\sigma\tau\lambda_2=\rho\lambda_2\le2\bar{\rho},\sigma\lambda_k\ge2\bar{\sigma}$, $\rho\lambda_k\le2\bar{\rho}$. Because by definition $\lambda_k\ge\lambda_2$ we have $W$ positive definite if $\sigma\lambda_2\ge2\bar{\sigma}$ and $\rho\lambda_k\le2\bar{\rho}$, i.e. $(\sigma,\tau)\in\mathcal{S}_2^*\cap\mathcal{S}_k^*$. Thus the $k$ coupled systems globally synchronize if $(\sigma,\tau)\in\mathcal{S}_2^*\cap\mathcal{S}_k^*\cap\mathcal{S}_M$.

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